# Why rhythmic canons are interesting

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**Abstract**

The subject of rhythmic canons has been revivified by new concepts, coming from the field of music composition. The present article is a survey of the mathematical knowledge in this field, from the 1950's to state of the art results.

## 1 What is this all about ? Paving the way

### 1.1 Rhythmic canons and tiling

I was introduced to the fascinating subject of rhythmic canons by people working at the Ircam, especially M. ANDREATTA and later on T. JOHNSON.

On closer investigation, this is the meeting point of numerous musical and mathematical issues, from spectral theory (FOURIER transforms and HILBERT spaces) (Lagarias and Wang, 1996) to factorisations of abelian groups (de Bruijn, 1955), from mosaics and tilings (like the azulejos in the Alhambra mirrored in DEBUSSY's *La Puerta del Vino* (Amiot, 1991)) to Galois theory and Galois groups on finite fields.

There are already many different and conflicting definitions for rhythmic canons (see (Mazzola, 2002, p 380-382), (Vuza, 1990-91), (Fripertinger, 2001) for instance). Our definition of a rhythmic canon will stress the regularity of the overall beat, allowing to work with integers.

**Definition** A rhythmic canon is a TILE (a purely rhythmic motif) repeated in several VOICES (for instance with several different instruments) with different OFFBEATS, so that TWO DISTINCT NOTES NEVER FALL ON THE SAME BEAT.

This is the musically intuitive definition. To be more precise, let us modelise it with two sets of integers: let $A$ be the the set of beats of the rhythmic motif, and $B$ the sets of the offbeats (meaning the beats where a new voice starts).

**Definition 1** *If two subsets of the integers, $A, B \subset \mathbb{Z}$ have the property that the map*

$$A \times B \ni (a, b) \mapsto a + b \in A + B$$

*is one to one, then we shall write $A + B = A \oplus B$ and call this sum direct.*

**Definition 2** *We have a rhythmic canon with motif (or »inner rhythm«) $A$ and set of entries (or »outer rhythm«) $B$ when $A, B$ are subsets of $\mathbb{Z}$ such that $A$ is finite and the sum $A + B$ is direct.*

Figure 1: a rhythmic canon with five voices

Thus $A$ is *tiling* $A \oplus B$. Without stress on $B$, we will simply say that $A$ *tiles*.

This definition is already quite restrictive, as it forbids two different voices to play on the same beat (which happens often in musical canons such as *the Musical Offering*). The restriction of a common pulsation however, which allows to rescale to integers, is not as stringent as it seems as proved in (Lagarias and Wang, 1996).

An example of a rhythmic canon is given by $A = \{0, 1, 4, 5\}$, the inner rhythm, and $B = \{0, 6, 8, 14, 16\}$, the outer rhythm of the canon. We also use a second description writing the beats of one voice from left to right in one line, where 1's stand for beats and 0's stand for silences, and different voices in consecutive lines vertically aligned.

```
110011000
      110011000
        110011000
              110011000
                110011000
```

Here is a graphic representation of the same canon, with a black square at every beat and time flowing from left to right.

A third equivalent definition, very useful as we will see, uses polynomials: it is customary to introduce a generating function $A(x)$ for a **finite** subset $A$ of $\mathbb{N}$ as follows:

Let $A(x) = \sum_{k \in A} x^k$ and *simile* for $B$; as a translate of rhythmic motif $A$ is obtained by multiplicating the associated polynomial $A(x)$ by a $x^k$, it is easily seen that

**Proposition 1** *For $A, B \subset \mathbb{N}$ both finite, we have a canon with inner and outer rhythm $A, B$ iff $A(x) \times B(x)$ is a polynomial with only 0's or 1's for coefficients.*

Up to a change of the time origin, it can be assumed that both sets $A$ and $B$ begin with 0 (i.e. $A(0) = B(0) = 1$ for the associated polynomials). This we will assume throughout the paper.

Such polynomials are called 0-1 polynomials and may viewed as elements of several rings (see section **4.2**).

The limitation on the finiteness of $A$ and $B$ will be discussed below (see Theorem 2 and Theorem 3).

The number of non zero coefficients in $A(x) \times B(x)$ will be exactly the product of the number of non zero coefficients in $A(x)$ and $B(x)$. These numbers are the cardinalities of the sets $A \oplus B, A, B$ respectively. Indeed in those conditions we compute easily the cardinality of the set $A$ by plugging 1 as the value of $x$ in $A(x)$:

**Lemma 1** $|A| = A(1)$

In the example above,

$$A(x) = 1+x+x^4+x^5 \quad B(x) = 1+x^6+x^8+x^{14}+x^{16} \quad A(x)\times B(x) = 1+x+\sum_{k=4}^{21} x^k$$

In this paper, we will usually try to get a set $A \oplus B$ without gaps. More precisely, essentially two cases arise:

## 1.2 Loops and lines

• In the last example, like in a fuga, there are some gaps in the beginning, but after a time several voices play together without gaps nor double beats, and this could be carried on for as long as wished. Most canons in the musical tradition similarly »tile a LOOP [1]«, meaning that it takes a while to get all voices singing together.

Thus we venture to introduce a generalization in our mathematical model, extending this tiling to the whole set $\mathbb{Z}$ of integers.

The above example can be rearranged to something simpler, called »tiling a line« by T. JOHNSON[2], namely

$$\{0, 1, 4, 5\} \oplus \{0, 2\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

So here is a more sophisticated example: $\{0, 1, 5\} \oplus \{0, 6, 9, 12\}$ may be extended to to $\{0, 1, 5\} \oplus \{\ldots, -3, 0, 3, 6, 9, \ldots, 8k-2, 8k, \ldots\}$.

As we will see in Theorem 3, if a finite rhythmic motif $A$ tiles the set of all integers then

$$\mathbb{Z} = (A \oplus B) \oplus n\mathbb{Z}$$

for some finite $B$, where $n$ is a period of the tiling. This means that $A \oplus B$ gives a complete set of residues modulo $n$, hence reduction modulo $n$ yields $\pi_n(A) \oplus \pi_n(B) = \mathbb{Z}/n\mathbb{Z}$ where we denote by $\pi_n(A)$ the elements of $A$ taken modulo $n$.

The smallest such $n > 1$ will be called the period of the canon.

This also shows that the reduction to finite $A$ *AND* $B$ yields no loss of generality and entails our most refined definition of a rhythmic canon:

**Proposition 2** *The set $A \subset \mathbb{Z}$ tiles an $n-$loop (a loop with period $n$) iff there exists a set $B$ with $\pi_n(A) \oplus \pi_n(B) = \mathbb{Z}/n\mathbb{Z}$.*

When the context is clear, we will drop the map $\pi_n$. The periodic character of such a tiling allows a cyclic representation (fig. 2):

Indeed it may even be best to set the representation of the canon on a torus (fig. 3), though perhaps this is a bit far from musical perception.

• Tiling a LINE in the sense used by composer T. JOHNSON (Johnson, 2001) means tiling a range of consecutive numbers. It is a rhythmic canon with a beginning and an end, without holes or double beats:

**Definition 3** *The set $A \subset \mathbb{Z}$ tiles a line iff there exists a set $B$ and $n \in \mathbb{N}^*$ with $A \oplus B = \{0, 1, 2, \ldots, n-1\}$.*

In our example, this would be the case with $B' = \{0, 2, 8, 10\}$ for instance.

Now obviously

**Theorem 1** *A tiling of a line gives a tiling of a loop, i.e. tiling a range $\{0, 1, 2, \ldots, n-1\}$ enables to tile both $\mathbb{N}$ and $\mathbb{Z}$.*

1 as coined by T. JOHNSON.

2 We will avoid this phrasing because the generally accepted mathematical meaning is different. See below for definition.

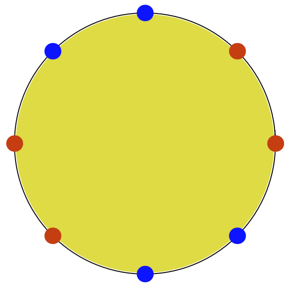

Figure 2: A tiling mod. 8

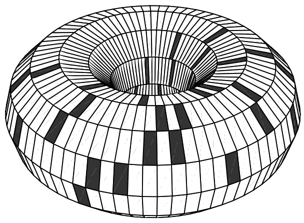

Figure 3: 12 voices modulo 72, on a torus

It is just a matter of repeating the tiling *ad infinitum*.

As the tiling of a line is a special case of tiling a loop, the same Theorem 3 will show that this is exactly the general problem of tiling $\mathbb{N}$ (with a finite tile).

But Theorem 6 will show that tilings of a line are much more repetitive than tilings of a loop.

An important remark: from now on, we have a duality because in

$$(A \oplus B) \oplus n\mathbb{Z} = \mathbb{Z}, \qquad (A \oplus B) = (B \oplus A)$$

both $A$ and $B$ are finite and play symmetric roles. We will call $B \oplus A$ the dual canon of $A \oplus B$.

## 1.3 Aperiodic canons

It is very easy to make very repetitive canons, by playing a periodic voice:

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
```

and stuffing it with copies of itself, like the following:

```
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
  1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
    1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
      1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
```

Indeed this happens quite frequently in »real« music. But we would like to know if less regular canons are possible.

So we will give a precise meaning to »avoid too much regularity in a rhythmic canons«:

- avoiding too regular entries of voices and
- avoiding a rhythmic pattern with a shorter period

As it happens, this connects directly to an old problem of decomposing $\mathbb{Z}$ and more generally abelian groups into a direct sum : see Theorem 5 below. As mentioned above, from Theorem 3, the most general rhythmic canon tiling with a finite motif is equivalent to a decomposition in direct sum of the cyclic group of order $n$:

$$A' \oplus B' = \mathbb{Z}/n\mathbb{Z}$$

The condition of non regularity means that there does NOT exist $p \in \mathbb{Z}/n\mathbb{Z}$ with $p \neq 0$ and

$$A' + p = A' \pmod n \qquad \text{or} \qquad B' + p = B' \pmod n$$

The existence of such decompositions is non trivial, the first historically found had a period of 108 and the smallest possible period is 72 (cf. (Andreatta, 1997)), so any examples will have to be pretty long to write down.

We will now state a number of mathematical results. It was surprisingly hard to unravel some of these, in old papers ridden with errors and missing proofs.

I will not reproduce here all proofs (see bibliography for these, especially (Sands, 1962)) but just the main ideas, especially insofar as musical concepts occur.

A good historical review can be found in (Tijdeman, 1995), with extensive bibliography.

# 2 Older results

## 2.1 Why infinite tiles are less interesting

We show in this paragraph why the restriction to finite motives is necessary, not only from obvious musical reasons, but also on the grounds of mathematical interest.

The program listed in the Appendix demonstrates a Theorem (Swenson, 1974) which shows that the most general tilings of $\mathbb{Z}$ can literally contain anything:

**Theorem 2** *Any direct sum of two finite sets (of integers) can be extended to an infinite direct sum decomposition of $\mathbb{Z}$:*

*If $A \subset \mathbb{Z}$ and $B \subset \mathbb{Z}$ are finite and $A + B = A \oplus B$, then there exist $A' \supset A,\ B' \supset B$ with $\mathbb{Z} = A' \oplus B'$.*

The next figure shows a few steps of such an extension. The gray region at each step outlines the preceding canon.

This result vindicates our choice of studying only finite motives, but also means that a given finite canon can be extended to span arbitrary large periods of time (though only by increasing the length of the motif AND the number of voices). Simulations do seem to grow in size quite fast, but with suitable truncating it might help to build some pretty musical (or pictural) compositions.

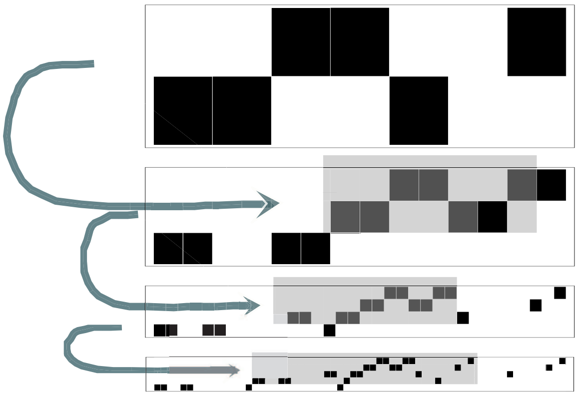

Figure 4: Swenson's theorem : expanding a canon

## 2.2 Repetition

The most musically appealing of all these results is the following one: any canon repeats itself.

**Theorem 3** *Any tiling of $\mathbb{Z}$ (by a* ***finite*** *tile) is periodic for some period $n$, meaning $\mathbb{Z} = A \oplus B$ boils down to $\mathbb{Z} = A \oplus (C \oplus n\mathbb{Z})$ for some $n$.*

The proof in (Hajòs, 1950) is obscure, N. DE BRUIJN had one around the same time, but it was not published, and Tijdeman (1995, p. 265)'s is a bit technical but it is the most convincing.

I will not reproduce here any of these proofs but only give their gist, which is two fold:

- Let $m = \max A - \min A + 1$ be the width of motif $A$, and consider time spans of width $m$. There is only a finite number of possible sequences of entries in such a period of time (at each beat, either a new voice begins, or it does not), at most $2^m$ certainly. So in a large enough collection of time spans of width $m$ beginning at different beats, there must be at least two identical sequences of entries by the pigeonhole principle.

  *In other words, there is but a finite number of possible combinations.*

- The next point is to prove that such a large sequence determines the whole tiling of $\mathbb{Z}$. The fact that the sequence is wider than the whole motif can be used to prove that there is only way to plug in the next voices.

  *In terms of orchestration, an idle musician is always waiting for the next gap to get in.*

- Now if the construction is the same starting at two different places, and wholy determined by what occurs in such a sequence, it means that the whole phenomenon is periodic (up to the difference of time between the two sequences).

  *Again in musical terms, when you are recognizing something you have heard before, it means that a repetition is taking place.*

Figure 5: Random concatenations of these two tiles give aperiodic tilings

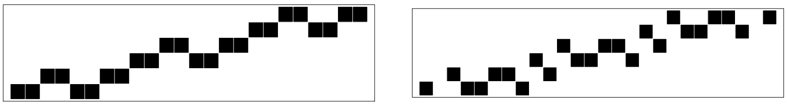

Figure 6: a canon, and one of its affine transforms

We must stress this is particular to tilings of $\mathbb{Z}$ with one tile. So it is useless to look for 'random' repetitions of one single motif, though it is possible with one motif and its retrogradation: (Lagarias and Wang, 1996) mentions $\{0, 1, 5\}$ which, together with $\{0, 4, 5\}$, allows any arrangement of the two 9-tiles on fig. 5,

which had been found also by T. JOHNSON by systematic search of short tilings with several 3-tiles.

To this day, little is known about tilings under a group of translations and central symmetries, even in the one dimensional case.

## 2.3 Affine transformation within a fixed period

It is possible to expand the rhythmic intervals in a motif without changing the period: the process has often been applied by composers, be it in the time or the pitch space (for instance Alban BERG's multiplies the basic *Lulu*'s serie by 5 to get Scholl's), consisting of an affine transform with a ratio relatively prime to the period.

For instance the tile $A = \{0, 1, 4, 5\}$ becomes $\{0, 3, 4, 7\}$ when multiplied by $3 \pmod 8$. It still tiles with the same $B$, taken to be $\{0, 2, 8, 10, 16, 18\}$ in fig. 6.

Musically it means it is possible to change the rhythmic motif without changing the table of entries of the different voices – or the reverse, as $A$ and $B$ play symmetric roles.

This kind of transform applies readily to the category of rhythmic canons. Indeed D.T. VUZA (Vuza, 1990-91, part 3) and R. TIJDEMAN (Tijdeman, 1995) have independently proved what amounts to the following result :

**Theorem 4** *If $p$ is coprime to $n$ and $A \oplus B \oplus n\mathbb{Z} = \mathbb{Z}$ then also*

$$(pA) \oplus B \oplus n\mathbb{Z} = \mathbb{Z} \qquad A \oplus (pB) \oplus n\mathbb{Z} = \mathbb{Z}$$

This means that any affine transform (with the affine group modulo $n$) of a tile also tiles, with the same period (indeed with the same outer rhythm). Thus we have a classification of rhythmic canons with fewer classes, up to independent affine transform of the inner or of the outer rhythm. Finally, this opens intriguing possibilities of a generative canon theory.

## 2.4 Hajòs groups and Vuza canons

The question of rhythmic canons with aperiodic inner and outer rhythms boils down to a factorisation of $\mathbb{Z}/n\mathbb{Z}$ with non periodic factors.

As it happens, it was noticed by G. HAJÒS and some others (in several steps and over several years in the 1950's) that most cyclic groups are HAJÒS groups in the following sense:

**Definition 4** *$\mathbb{Z}/n\mathbb{Z}$ is a* HAJÒS *group if, for any decomposition $A \oplus B = \mathbb{Z}/n\mathbb{Z}$ there exists $p \notin n\mathbb{Z}$ with $A + p = A \pmod n$ or $B + p = B \pmod n$.*

Non-HAJÒS groups are sometimes called »bad groups« and admit to several interesting generalizations, irrelevant here; see (Sands, 1962) or (Tijdeman, 1995) for a general discussion of factorisations of finite abelian groups..

The following theorem was rediscovered independently by Dan Tudor VUZA around 1990 (Vuza, 1990-91):

**Theorem 5** *$\mathbb{Z}/n\mathbb{Z}$ is a bad group, i.e. a non* HAJÒS *group (i.e. there exists an aperiodic decomposition of $\mathbb{Z}/n\mathbb{Z}$) in the following cases, and in these cases only: when $n = p^\alpha$ or $n = p^\alpha q, n = p^2q^2, n = p^\alpha qr, n = pqrs$, $p, q, r, s$ being primes.*

D.T. VUZA called the aperiodic canons corresponding to such decompositions »Regular Canons of Maximal Category«, and he built up an algorithm for producing independently some inner and outer aperiodic rhythms for a »bad« $\mathbb{Z}/n\mathbb{Z}$.

The smallest $n$ for which $\mathbb{Z}/n\mathbb{Z}$ is bad is $n = p^2q^3, p = 3, q = 2$ i.e. $n = 72$. An example is given below. A good historic outline of this is (Andreatta, 1997). Strangely, the first historical example of a bad group was $\mathbb{Z}/108\mathbb{Z}$, though this group was forgotten by D.T. VUZA (and rightfully restored by M. ANDREATTA).

It is worthy of note that from the 36 canons given by D.T. VUZA's algorithm for $n = 72$, only 2 orbits remain under affine transformations (T. NOLL, H. FRIPERTINGER and others had looked into this before).

## 2.5 Reduction

In 2002 in Ircam(Amiot, 2002) I wondered if something alike to non-G. HAJOS groups did exist in the monoid $\mathbb{N}$, that is to say aperiodic tilings of a line, which would have been an even rarer material than aperiodic tilings of a loop. The answer stems from a rather hidden (though often alluded to) result:

**Definition 5** *The $m-$ zoom of the rhythmic canon with inner rhythm $A$ and outer rhythm $B$ is the canon with inner and outer rhythms $A', B'$ where in terms of the generating polynomials*

$$A'(x) = (1 + x + x^2 + \dots x^{m-1}).A(x^m) \qquad B'(x) = B(x^m)$$

Meaning musically that:

- to get $A'$, each note (resp. each silence) in $A$ is replaced by $m$ consecutive notes (resp. silences)
- to get $B'$, the metronomic tempo is multiplicated by $m$: for instance for $m = 2$, a beat in quarter notes should be replaced by a beat in eighth notes.

**Theorem 6** *[N.* DE BRUIJN*] Every tiling (by a finite tile) of $\mathbb{N}$ is reducible to a smaller tiling, i.e. is an $m-$zoom of a smaller tiling for some $m$.*

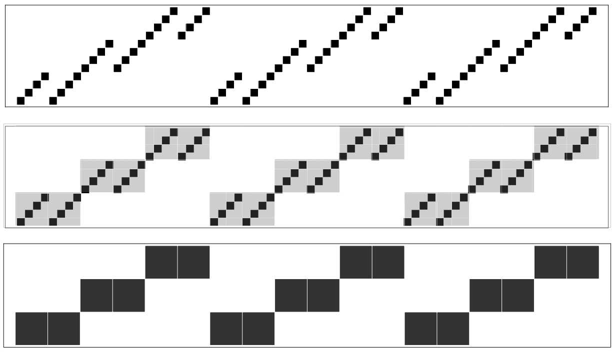

Figure 7: zooming out, dividing the tempo and number of voices by 1/4

The demonstration (by induction) is a lemma (de Bruijn, 1955) in a paper on British number systems [sic]. Several years elapsed before the relevance of this lemma to factorisation of semi-groups was noted.

Hence, as any tiling of $\mathbb{N}$ is periodic (a combination of Theorem 3 and Theorem 1), the above Theorem reads

*Every tiling of a finite range* $\{0, 2, \ldots, n-1\}$ *is reducible to a smaller tiling.*

This means that a canon »tiling a line« in T. JOHNSON's sense (with only translations of only one tile) must have - mathematically - a very simple structure. Indeed it could be built up from scratch (that is to say from one note, played once) and recursively replacing

- one note by a succession of $p$ notes in the same voice, or
- one voice by a succession of $p$ voices (using the duality between inner and outer rhythm)

with arbitrary $p$ each time.

Also it enables to make greater canons from small ones, which has interesting abstract as well as practical consequences: computerized tools for such transformations of canons are currently under development.

## 2.6 Equirepartition

Of course if $\mathbb{N}$ (or $\mathbb{Z}$) can be written $A \oplus B \oplus n\mathbb{Z}$ it means that $A \oplus B$ is a complete set of residues modulo $n$, or in other words there is equirepartition modulo $n$; in polynomials this is expressed by:

$$A(x) \times B(x) \equiv 1 + x + x^2 + \ldots + x^{n-1} \pmod{x^n - 1}$$

Indeed this equirepartition result can be reversed to build up trivial (but still musically interesting) canons:

Start from $A(x) = \Delta_n(x) := 1 + x + x^2 + \ldots + x^{n-1}$, a trivial tile, and randomly add multiples of $n$ to each the exponents above, e.g. get

$$A'(x) = 1 + x^{1+n\,r_1} + x^{2+n\,r_2} + \ldots + x^{n-1+n\,r_{n-1}}$$

which tiles trivially as $A' \oplus n\mathbb{Z} = \mathbb{Z}$. Indeed a reverse is true :

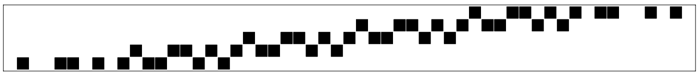

Figure 8: Example of equirepartition mod. 9

**Theorem 7** *A finite subset $A'$ of $\mathbb{N}$, beginning with 0, tiles $\mathbb{Z}$ trivially with period $n$ – meaning $A' \oplus n\mathbb{Z} = \mathbb{Z}$ – iff $A'(x) = 1 + x^{1+n\,r_1} + x^{2+n\,r_2} + \ldots + x^{n-1+n\,r_{n-1}}$.*

This could be called a trivial canon, for classification's sake: the period of the canon is really the number of notes of the motif.

Indeed it is easily seen that $\mathbb{Z} = \bigcup_{k\in\mathbb{Z}} (A' + n\,k)$ ($\bigcup$ denotes here a partition of $\mathbb{Z}$ into disjoint subsets). So $A'$ tiles, as $A' \oplus n\mathbb{Z} = \mathbb{Z}$.

It must be stressed that this NOT the general case, indeed this is what G. HAJOS (or D.T. VUZA)'s theorems are about. But

**Theorem 8 (N. DE BRUIJN)** *When a tile is of prime size, $|A| = p$, then $A$ is of the form above, i.e.*

$$A(x) \equiv 1 + x + x^2 + \ldots + x^{p-1} \pmod{x^p - 1}$$

The proof is not too difficult (using Lemma 2 below, $A$ must have a $\Phi_{p^\alpha}$ as a factor and reduction $\pmod{x^p - 1}$ leaves no choice but $\alpha = 1$). See also Theorem 9 (Newman, 1977), extending this to $n = p^\alpha$ with some complications.

These questions of equirepartition bring to mind a number of fascinating related issues, among which FOURIER analysis.

Indeed several tiling problems in finite dimension have led to the so-called SPECTRAL CONJECTURE [FUGLEDE, 74] well explained in (Laba, 2000), where regularity in a tiling is equivalent to exhibiting a HILBERT base of a function space on a tile. To state it more precisely:

**Spectral Set Conjecture 1** *A region $T$ tiles $\mathbb{R}^n$ (by translations) if and only if there is a set of exponentials*

$$\mathcal{S} = \{e_\lambda \mid \lambda \in \Lambda\} \qquad e_\lambda : x \mapsto \exp(2i\pi\lambda.x)$$

*whose restrictions to $T$ are a HILBERT base of $L^2(T)$.*

To give the simplest example, $T = [0, 1[$ tiles $\mathbb{Z}$ and the $e_n : t \mapsto e^{2i\pi\,n\,t}, n \in \mathbb{Z}$ are a HILBERT base of 1-periodic functions. Apparently very slow progress has been made in that direction of late.

# 3 Recent results

## 3.1 Around cyclotomic polynomials

### 3.1.1 A useful tool: the $\Phi_n$

From now on, we will make heavy use of the polynomials associated with finite subsets of $\mathbb{N}$.

For the sake of clarity, I recall the definition of a polynomial associated with a subset of $\mathbb{N}$:

$$A(x) = \sum_{i \in A} x^i$$

remembering that up to a change of the time origin it it always possible to assume that in the canon with inner and outer rhythms $A, B$, one has

$$\min A = \min B = 0$$

This will be assumed throughout.

I will begin with some very simple lemmas (not stated in the recent technical papers) without which the greater theorems remain cryptic.

Most properties of cyclotomic polynomials are to be found in any good textbook on algebra (*Algebra*, by S. Lang, for instance).

**Definition 6** *The $n^{th}$ cyclotomic polynomial is*

$$\Phi_n(x) = \prod_{\gcd(k,n)=1} (x - e^{2i\,k\,\pi/n}) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)}$$

*where $\mu$ is the* MÖBIUS *function.*

This the monic polynomial whose roots are the primitive units of order $n$, or equivalently the minimal polynomial of one such root.

Classically they are the irreducible factors of $x^n - 1$:

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \qquad \Delta_n(x) := 1 + x + x^2 + \ldots + x^{n-1} = \prod_{d|n, d \neq 1} \Phi_d(x)$$

Usually $\Phi_n$ have coefficients which are $0, 1$ or $-1$. We will need:

**Lemma 2** $\Phi_n(1) = 1$ *if and only if n is NOT a power of a prime number. Conversely, $\Phi_{p^\alpha}(1) = p$.*

This follows by induction from the preceding expression and the formula

$$\Phi_{p^\alpha}(x) = \sum_{k=0}^{p-1} x^{k\,p^{\alpha-1}} = \Phi_p(x^{p-1})$$

For instance, $\Phi_8(x) = 1 + x^4$.

The importance of these particular polynomials lies in

**Lemma 3** *If $A \oplus B \oplus n\mathbb{Z} = \mathbb{Z}$, then for all $d \mid n$ ($d \neq 1$) $\Phi_d$ is a divisor of either $A(x)$ or $B(x)$.*

The assertion reads

$$A(x) \times B(x) \equiv 1 + x + x^2 + \ldots + x^{n-1} \pmod{x^n - 1}$$

Thus $\Delta_n$, which divides itself and hence $x^n - 1 = \Delta_n \times (x - 1)$, must be a divisor of $A(x) \times B(x) = \Delta_n + Q(x) \times (x^n - 1)$; the cyclotomic factors of $\Delta_n$, being irreducible, must divide either $A$ or $B$.

### 3.1.2 Canonic and polycanonic rhythms

One main open problem (for musicians) is : can I make a rhythmic canon with this given (finite) tile (i.e. inner rhythm) ?

Up to 1998 and a paper by (Coven and Meyerowitz, 1999) it was still unknown what one could get, even with as simple as a 6 notes motif. Only the case $|A| = p^\alpha$ was known (Newman, 1977).

I could not find NEWMAN's paper, but with the ideas below I could get an idea of the results: something like the following

**Theorem 9** *If $A$ tiles and $|A| = p^\alpha$, where $p$ is a prime and $\alpha \in \mathbb{N}^*$, then for adequate $\beta, k \in \mathbb{N}^*$*

$$A \equiv k\Delta_{p^\beta} \pmod{X^{p^\beta} - 1}$$

*i.e. there is equirepartition modulo $p^\beta$.*

Equirepartition modulo the size of $|A|$ is not mandatory, as $A$ may have factors $\Phi_{p^\beta}$ with $\beta \neq \alpha$ as seen in this example with overall period $n = 8$:

$$A = \{0, 1, 4, 5\} \qquad |A| = 4 \qquad B = \{0, 2\} \qquad A(x) \equiv 2(1+x) = 2\Delta_2(x) \pmod{x^2-1}$$

Some progress has been made on the case $n = p^\alpha.q^\beta$ and partial progress for $n = p^\alpha.q^\beta.r^\gamma$. The following criteria make heavy use of cyclotomic factors of the polynomials associated with the tiling, essentially showing that there are mostly cyclotomic polynomials as factors of $A$ and $B$, with a rigid arrangement.

### 3.1.3 Condition $(T1)$

Before stating the condition we explain what it is about. Say $A$ is tiling $\mathbb{Z}$. The period of the tiling must be a multiple of $|A|$ (as $n = |A|.|B|$). Moreover, any cyclotomic $\Phi_d$ dividing $A(x)$ will divide $A(x) \times B(x)$ and $x^n - 1$, hence $\Phi_d(x) \mid A(x) \Rightarrow d \mid n$.

A number of cyclotomic polynomials may divide $A(x)$ (for all $1 \neq d \mid n$ in the trivial case $A = \{0, 1, 2, \ldots, n-1\}$). Now by Lemma 2, only the few of these for which $d$ is a prime power will contribute to the value of $A(1)$ (remember this is the cardinality of the set $A$). But according to Lemma 3, all the prime powers $p^\alpha \mid n$ will be there and every one will contribute for the value $\Phi_{p^\alpha}(1) = p$. Thus $p$ occurs at least $m(p)$ times (multiplicity of $p$ in $n$, and also the number of distinct powers of $p$ dividing $n$) in $A(1)$ or $B(1)$. These being integers, with $A(1).B(1) = n = \prod_{p|n} p^{m(p)}$, it means that all non-cyclotomic factors contribute nought to the values of $A(1)$ or $B(1)$ ! We have just proved Theorem $(B1)$ of (Coven and Meyerowitz, 1999) (this is true for $A$ and $B$) :

**Definition 7** *For $A \subset \mathbb{N}$, we set $S_A$ = the set of prime powers $p^\alpha$ with $\Phi_{p^\alpha}$ dividing $A(x)$.*

**Theorem 10** *If $A$ (finite subset of $\mathbb{N}$, beginning with 0) tiles $\mathbb{Z}$, then*

$$(T1) \qquad A(1) = \prod_{p^\alpha \in S(A)} \Phi_{p^\alpha}(1)$$

This condition is necessary, but not sufficient. Though rather obvious, it was apparently not spotted before 1998.

### 3.1.4 Condition $(T2)$

- Condition $(T1)$ explains what happens to » visible « cyclotomic factors.
- Condition $(T2)$ tells where some ot the » invisible « cyclotomic factors (those factors with value 1 in 0) must lie:

  $(T2)$ If $p^\alpha, q^\beta, \ldots, r^\gamma \in S_A$ are powers of different primes $p, q, \ldots, r$, then $\Phi_{p^\alpha q^\beta \ldots r^\gamma}$ is a divisor of $A(x)$.

This is a kind of stability property for the set $\mathcal{D}(A) = \{d, \Phi_d \mid A(x)\}$.

So we have several factors of $A(x)$:

- The $\Phi_{p^\alpha}$, subject to condition $(T1)$.
- The $\Phi_{p^\alpha q^\beta \ldots r^\gamma}$ with $p^\alpha, q^\beta, \ldots r^\gamma \in S_A$: this is related to condition $(T2)$.
- Compound cyclotomic factors $\Phi_{a.b\ldots}$ with $a \in S_A, b \in S_B$,which are factors of either $A(x)$ or $B(x)$.
- Non cyclotomic factors (scarce).

Recently E. COVEN and A. MEYEROWITZ(Coven and Meyerowitz, 1999) proved the following rather easy theorem:

**Theorem 11** *If A satisfies $(T1)$ and $(T2)$, then A tiles.*

The proof uses only elementary facts about cyclotomic polynomials and their products.

They also prove a more difficult theorem, using essentially the important Theorem 4:

**Theorem 12** *If $A$ tiles AND $|A|$ has only two prime factors, then $A$ satisfies $(T1)$ and $(T2)$.*

So $(T1)$ + $(T2)$ is sufficient for $A$ to tile in any case, and also necessary for »simple« values of $|A|$. Maybe this a necessary and sufficient condition for any $|A|$, with any number of prime factors: no counter example is yet known.

This would be a substantial part of a proof of the spectral hypothesis for (finite) tilings of $\mathbb{Z}$, according to (Laba, 2000).

### 3.1.5 An example and an answer

To understand the conditions $(T1)$ and $(T2)$ above, let us look at a random example, given by VUZA's algorithm (Vuza, 1990-91) for constructing aperiodic canons:

$$A = \{10, 18, 26, 28, 36, 44\} \qquad B = \{12, 18, 19, 23, 24, 43, 47, 48, 54, 60, 67, 71\}$$

We get the following factorizations with all factors being cyclotomic, except one ($A$ and $B$ have been translated so as to begin with 0 as usual):

$$A(x) = \Phi_4\Phi_6\Phi_3\Phi_{12}\Phi_{24}\Phi_{36}$$

$$B(x) = \Phi_2\Phi_8\Phi_9\Phi_{18}\Phi_{72}\times(1-X+X^2-X^3+X^7-X^{13}+X^{14}-X^{15}+X^{16}-X^{17}+X^{18})$$

We get $S_A = \{4, 3\}$ and $S_B = \{2, 8, 9\}$. Thus:

- Condition $(T1)$ for $A$ reads $A(1) = 2 \times 3 = 6$, and for $B$ it is $B(1) = 2 \times 2 \times 3 = 12$.

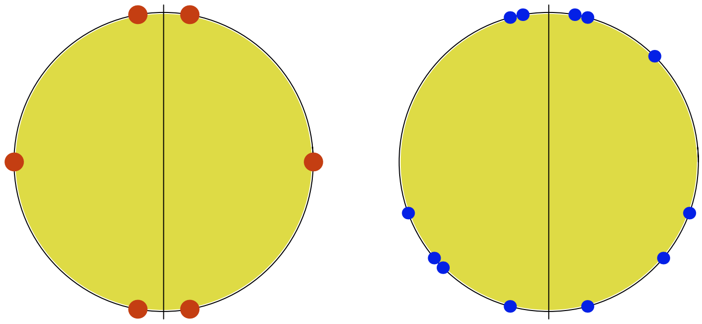

Figure 9: $A$ is an exact palindrom $B$ almost is

- Condition $(T2)$ for $B$ means that factors $\Phi_{2\times 9}$ and $\Phi_{8\times 9}$ are in $B(x)$, which is true. Similarly for $A$.

As was first noted by (Andreatta, 1997), $A$ is a perfect palindrom, $B$ almost but not quite, as seen on the following picture:

(Andreatta, 1997) first asked whether this is general and why. Though of course it is difficult to define rigourously such an elusive quality as »almost palindromic«, I think the theorems in (Coven and Meyerowitz, 1999) help to understand why this is so.

A motif is palindromic when its polynomial is equal to its reciprocal: say

$$P(x) = x^d P(1/x) \qquad \text{where } d \text{ is the degree of } P$$

But from their definition, all cyclotomic polynomials are palindromic: if $\xi$ is a $n^{\text{th}}$ root of unity, so is $1/\xi$. Hence any product of cyclotomic polynomials is palindromic, and the above theorems tell us that in a rhythmic canon the factors of the associated polynomials are mostly cyclotomic. So these factors are (almost) palindromic, which is in my opinion as close as one could get to an answer to M. ANDREATTA's question.

The Coven-Meyerowitz theorems are very gratifying, especially if Theorem 12 is eventually proved for any period.

What they leave in the dark are the following points:

- What are the NON cyclotomic factors doing ?
- How are the 'neutral' cyclotomic polynomials (not in $(T1)$ or $(T2)$) dispatched between $A$ or $B$, which is probably linked to the major question:
- How does one ensure that a product of such polynomials has only 0 or 1 as coefficients ?

Three years later, GRANVILLE, LABA,& WANG managed partially the case of 3 prime factors (Granville et al., 2001):

**Theorem 13** *If $A \oplus B = \mathbb{Z}/n\mathbb{Z}$, $|A| = p^\alpha q^\beta r^\gamma$, $|B| = pqr$; if $\Phi_p, \Phi_q, \Phi_r$ all are factors of $A(x)$, then so are $\Phi_{pq}, \Phi_{rq}, \Phi_{rp}$.*

The proof takes 15 pages of heavy calculations. The number 15 plays a lighter part in the last section of this paper.

# 4 Polyrhythmic canons and future results

Though many questions remain open concerning canons with ONE motif, still less is known when several motives (tiles) are allowed, even with just a motif and its reverse. But perhaps this greater complexity paves the way for new tools and deeper results on old questions.

## 4.1 Johnson's question and the number 15

About one year ago in Royan at the JIM (Johnson, 2001), T. JOHNSON began to try tiling with a motif : $\{0, 1, 4\}$, and its AUGMENTATIONS. To continue working with integers, he selected augmentations by 2: $2\times\{0, 1, 4\} = \{0, 2, 8\}$, $4 \times \{0, 1, 4\} = \{0, 4, 16\}$ and so on.

Setting $J(x) = 1 + x + x^4$, the augmentations read $J(x^2) = 1 + x^2 + x^8$, $J(x^4) = 1 + x^4 + x^{16} \dots$, and the problem is to find 0-1 polynomials satisfying

$$A(X) \times J(X) + B(X) \times J(X^2) \quad [+ C(X) \times J(X^4) + \dots] = \Delta_n[X] \qquad (1)$$

It is easy enough to find by hand the smallest tiling, of period 15:

$$(1 + X^2 + X^8 + X^{10}) \times J(X) + X^2 \times J(X^2) = \Delta_{15}(X)$$

Shortly after Tom stated his problem, A. TANGIAN came up with a Fortran program and a list of solutions up to a given size (Tangian, 2001).

There was one salient fact: all periods were multiples of 15, 1 for length 15, 6 for length 30 (beginning with $J$), a.s.o. Tom asked if this was general and could be proved.

## 4.2 A tiny step in an unexpected direction

When I first tried tackling Tom's problem I wrongly considered GALOIS groups of cyclotomic fields over the rationals. Fortunately this was avoidable, and a much simpler idea worked: if such polynomial identities in $\mathbb{Z}[X]$ represent tilings, they are also identities in the ring of polynomials with coefficients in any field (as the only numbers involved in the calculations are 0's and 1's): see Theorem 14 below.

The first idea that springs to mind is to study the case of the smallest field, with two elements, e.g. $GF[2] = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2 = \{0, 1\}$. The correspondance with the set of 0-1 polynomials is the closest possible: it is one to one and onto, though unfortunately the latter set is not closed under either addition or multiplication.

Indeed such identities in $\mathbb{F}_2[X]$ stay true when expanding the field to any field with characteristic 2.

Now computing relation 1 in such a field with elements of (multiplicative) order 15 held the solution.

The smallest such field is $GF[2, 4] = \mathbb{F}_{16}$. It so happens it is the decomposition field of $J$ over $\mathbb{F}_2$: that means it it the cheapest way to get a root (indeed, four different roots) of $J$ (this is because $J$ is easily seen to be irreducible in $\mathbb{F}_2[X]$).

It so happens that any of these roots, say $r \in \mathbb{F}_{16}$, is exactly of order 15, meaning that $r^n = 1$ iff $n$ is a multiple of 15[3].

Furthermore, in $\mathbb{F}_2[x]$ one has $J(x^2) = \big(J(x)\big)^2$ (this is true for all polynomials over a field of characteristic 2 because in these fields »1+1=0 «). Hence $r$ is also a root of $J(x^2), J(x^4), \ldots$.

But then equation 1 yields

$$0 + 0\,[+0 \ldots] = 1 + r + r^2 + \ldots + r^{n-1} = (1 - r^n).(1 - r)^{-1}$$

So $r^n = 1$, hence $15 \mid n$, qed.

This does not always work: for the motif $\{0, 1, 3\}$ for instance, one only gets $7 \mid n$ by the same method. But as all tiles play three notes, a solution must have at least length $21 = 3 \times 7 \mid n$, as is indeed the case (the smallest solution is 21 long).

Worse still, this method provides only a necessary condition, which looks a long way from ascertaining whether a solution does exist !

And it is pretty difficult to haul solutions from similar equations in $\mathbb{F}_2[x]$ (which are numerous) back to $\mathbb{Z}[x]$. I could only prove a simple criteria, akin to $(T1)$:

**Theorem 14** *An identity of the kind $A(x) \times B(x) + C(x) \times D(x) + \ldots = W(x)$ [only sums and products] in $\mathbb{F}_2[x]$ also holds in $\mathbb{Z}[x]$ if, and only if, it holds when a non negative real number $\alpha > 0$ is substituted for $x$.*

Proof (sketched): otherwise cancellations occur, as in this example:

$(1+X+X^3)+(X+X^2+X^3) = 1+X^2$ in $\mathbb{F}_2[X]$ but not in $\mathbb{Z}[X]$ : indeed $3+3 > 2$!

This is a strange situation: we ponder in which cases the canonical injection $\mathbb{F}_2[X] \to \mathbb{Z}[X]$ works like a ring morphism !

Taking $\alpha = 1$ we are not far from condition $(T1)$.

Taking $\alpha = 2$ or higher, MERSENNE numbers and rep-units appear.

'Many identities in $\mathbb{F}_2[x]$ do not stand anymore when read in $\mathbb{Z}[x]$. But if an identity is true in a number of similar rings it will stand in $\mathbb{Z}[x]$:

**Theorem 15** *Consider an identity of the kind $A(x) \times B(x) + C(x) \times D(x) + \ldots = W(x)$ between polynomials with coefficients all 0 or 1. Then the following assertions are equivalent :*

- *It holds in $\mathbb{Z}[x]$;*
- *It holds in every $\mathbb{F}_p[x]$.*
- *It holds in every $k[x]$ ($k$ being any field).*

The only non trivial implication is $(ii) \to (i)$, which is already true if the identity stands for a large enough number of (prime) $p$'s, by multiple application of the chinese remainder theorem.

This is coherent with the YONEDA philosophy (see **??**]Mazzola) wherein something is true as soon as it is true for multiple partial viewpoints.

3 Using Lagrange's or Fermat's small theorem, it is easily seen that for any $r \in \mathbb{F}_{16}^*$, $r^{15} = 1$. It only remains to check that if $J(r) = 0$, $r$ cannot be of order 1,3 or 5.

### 4.3 Prospects

Many things should be attempted, and several might be affordable:

- Counting solutions of the different problems above.
- Exploring this new concept of polyrhythmic canons: this would be more difficult, but new ideas will arise, like GALOIS theory, which might help with the old questions (like » is $(T1) + (T2)$ necessary and sufficient ?«). Already H. FRIPERTINGER (Fripertinger, 2001) made some interesting remarks on generalizing the techniques used on the above problem.

  Specifically, rhythmic augmentations in terms of polynomials are obtained by a FROBENIUS automorphism, which means that GALOIS theory on finite fields is relevant.
- Another kind of polyrhythmic canon uses just a tile and its reverse. When is it possible to tile with these two ?
- It would be useful to have a data base of all canons (of given order), like SLOANE's famous encyclopedia of integer sequences online. For instance, it would help to
- Find all D.T. VUZA canons of given order (meaning aperiodic canons, not only the ones provided by his algorithm).
- Also it is necessary to develop a set of computer tools for all basic transforms on rhythmic canons: reductions to canonical (!) forms, expansion, reduction, affine transforms...This is currently under development and will eventually find its natural abode in environments like *OpenMusic* or *Rubato*.

I can only hope it will be noticed how a musician's look on these problems often helps to better understand the mathematics involved. Conversely, a musician's questions to the mathematician are so unexpected that they seldom fail to suggest solutions, if only to other problems.

The musical culture might help solve difficult questions; for instance, no really good upper bound for the period of a canon is known from the width of the motif ($m = \max A - \min A + 1$). As noticed by (Coven and Meyerowitz, 1999), the pigeonhole principle used in demonstration of Theorem 3 yields an upper bound of about $2^m$, though musical experience would suggest that $n \leqslant 2m$, but this is as yet a conjecture.

In an other direction, perhaps musical intuition might help to solve difficult problems, like the spectral conjecture (at least in dimension 1). To me it looks like musical considerations provide new ideas, concepts, and insights for non trivial mathematical problems.

I will thank again MMs M. ANDREATTA, T. JOHNSON, H. FRIPERTINGER, A. TANGIAN for prompting my interest in these fascinating matters, T. NOLL for useful Mathematica notebooks, the reviewers for very helpful advice, and last but not least, G. MAZZOLA for kindly inviting me to Mamuth.

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## Appendix : an implementation of Swenson's theorem with Mathematica™

The principle is as follows:

Given A,B in direct sum, take $g$ the smallest integer not in $A \oplus B$ (closest to origin on either side).
Example: $A = \{0, 1, 4\}, B = \{0, 2, 7\}, A \oplus B = \{0, 1, 2, 3, 4, 6, 7, 8, 11\}, g = -1$.

Now find by iteration the smallest $m$ such that $A \oplus \{m\}$ and $B \oplus \{g - m\}$ is a direct sum again. We may try first $m$ to be the smallest gap in $A$ and increment if it does not fit. When it does, we have expanded the direct sum (as $g$ will now be in it, it goes farther from origin) and so the process will eventually reach every integer, which is a constructive proof of Swenson's theorem.

```
oPlus[a_,b_] := Union[Flatten[Outer[Plus,a,b]]]
oPlusCheck[a_,b_] := Module[{op = oPlus[a,b]},
If [Length[a]*Length[a]== Length[op],op, {}]]

firstGap[mySet_] := Module[{m=0},
         While[MemberQ[mySet,m],
            If[m<=0,m=1-m,m=-m]]
          ;m]

enlargeCanon[A_, B_]:=
Module[{g=firstGap[oPlus[A,B]], m, Anew, Bnew, op},
(* declaring local variables *)

m = firstGap[Union[A, g-B] ];
op = {};

(* now we try different m until it fits *)

While[op=={},
Anew = Append[A,m];
Bnew = Append[B, g-m];
op = oPlusCheck[Anew, Bnew];
;  m++
];
(* should be OK by now,
we list the solutions after sorting them
*)
Sort /@ {Anew, Bnew }
]
```