TWO HEXACHORDAL THEOREMS IN GENERAL COMPACT GROUPS

Abstract : The hexachordal theorem is a classical and non obvious result in music theory, stated in most textbooks (see [4, 12, 11]*). In its diverse forms essentially states that a subset and its complementary subset have the same autocorrelation function : the map $iv(A): \quad t \mapsto \mu(A \cap (A+t))$ that measures the intersection of A with its translate by t is identical to iv for the complement of A (up to a constant in general, which is 0 when A has the same measure as its complement, as in the original 'hexachord' case in the cyclic group $\mathbb{Z}_{12}$). So far this has been proven in abelian compact topological groups. Here we state the generalization to non abelian compact groups, like the four dimensional sphere $S^3 = SU(2)$. As it happens there are two hexachordal theorems in this general case, not one.*

I. HISTORY OF HEXACHORDAL THEOREMS

The origin of the hexachordal theorem is somehow lost in the mists of time : according to Milton Babbitt, the first proof around 1960 involved very deep mathematics (though some suspect that this declaration is a joke). The context was the interval vector of subsets of the set of notes, later known as pc-sets. This map was defined on the cyclic group $\mathbb{Z}_{12}$ by

$$\forall A \subset \mathbb{Z}_{12} \quad iv(A): \quad t \mapsto \sharp(A \cap (A+t))$$

For instance when $A = \{0,1,4,6\}$ the map takes values $< 4,1,1,1,1,1,2,1,1,1,1,1 >$. The original Hexachordal Theorem states

Théorème 1. *For any hexachord $A \subset \mathbb{Z}_{12}$ (i.e. $\#A = 6$), $iv(A) = iv(\overline{A})$ where $\overline{A}$ denotes $\mathbb{Z}_{12} \setminus A$.*

The map iv can be seen as an autocorrelation function, or as as the histogram of the intervals between A and itself. A generalization states that $\forall\, A \subset \mathbb{Z}_{12}$, $iv(A) = iv(\overline{A}) + C$ where the constant C is easily computed for $t = 0$, since $iv(A)(0) = \#(A)$. The most general statement involves the original interval function introduced by Lewin in his very first paper : [8]

Théorème 2. *$IFunc(\overline{A}, \overline{B}) = IFunc(A, B) + C$ for any subsets A, B of a cyclic group $\mathbb{Z}_c$, where $IFunc(A, B)(t) = \#((A+t) \cap B)$.*

It is stated and proved in two lines in [12].
Now where next ? In [13] the simplest continuous case, the circle S^1, is presented, as a limit of the discrete cyclic groups $\mathbb{Z}_n$ when n grows to infinity. But in fact the theorem makes sense in any abelian group (denoted additively) with a Haar measure, i.e. with a map from its (measurable) subsets to $\mathbb{R}_+$ satisfying $\mu(A+t) = \mu(A)$. The author had noticed this fact when discussing with Jack Douthett the generalization of Maximally Even Sets (by way of Fourier Transform) to torii, first discrete and then the general $T^n = S_1 \times \ldots S_1$ with the unit product measure. In this extended form, it was first stated in print, as far as we know, by John Mandereau in his master thesis[10] :

Théorème 3. *Let $(G, +)$ be a topological compact group, abelian, and μ be a Haar measure on G. $\forall\, A \subset G$, define $iv(A): \quad t \mapsto \mu(A \cap (A+t))$. Then*

$$iv(A) = iv(\overline{A}) + \mu(A) - \mu(\overline{A})$$

(notice that if A is measurable then so are $A + t$ and $A \cap (A + t)$)
The problem with more general groups is that all of this does not make sense anymore : in particular there are two different actions of a group on its subsets, by left and by right translation. We still want our measure to be translation invariant, in order to define the map iv, and finite : else a finite measure subset would have an infinite complement. So the general definition is

Définition 1. *Let $iv_r(A)$ be the map*

$$g \mapsto \mu(A \cap (A \times g)) = \int_G 1_A(h) 1_A(h \times g^{-1})\, d\mu(h)$$

(written as an integral with μ a Haar measure on the compact group G). Similarly the left interval content is defined as

$$iv_l : \quad g \mapsto \mu(A \cap (g \times A)) = \int\limits_G 1_A(h) 1_A(g^{-1} \times h)\, d\,\mu(h)$$

Then we have the result of this paper :

Théorème 4. *Let G be a topological group, compact, separable (i.e. with a dense enumerable subset). Then G has a Haar measure μ, left and right invariant, unique up to a multiplicative constant, and for any (measurable) subset A the map $iv_r(A)$ coincides with $iv_r(\bar{A})$, up to a constant ; and similarly with iv_l as defined above.*

[13] asked whether the hexachordal theorem could be generalized from S^1 to the usual sphere S^2. But this is hopeless, as there is no natural way to define a correlation function for subsets of S^2. After S^1, the next sphere with a group structure is S^3, which will be used as an example in the following section. Other possibilities and extensions will be discussed in the last section.

II. Proofs and other technicalities

1. **Haar measure.** The main topological result is well known in group theory :

Théorème 5. *Let G be a topological group, compact, separable (i.e. with a dense enumerable subset). Then G has a Haar measure μ, left and right invariant, unique with the condition $\mu(G) = 1$.*

Preuve. Usually this is proved using the Kakutani fixed point theorem on the convex and compact (non empty !) set of probability measures on G. This can be found in any textbook on topological groups, see [6] for instance. □

Remarque 1. *This theorem is true for any compact Lie group, since as a submanifold of $\mathbb{R}^n$ it admits a dense enumerable subset. Also the differential structure enables to build in each point a multilinear (volume) form using the tangent bundle, which provides an invariant measure free of charge. This is true for all classical (linear) groups, rotation groups aso, and for S^3, the unit sphere in $\mathbb{R}^4$, which admits a canonical group structure as SU(2) (it is the universal covering of the rotation group $SO(3)$). We give some details on this case as an illustration of the theorem.*

2. **Haar measure on S^3.** We define $G = SU(2) = S^3$:

Définition 2. *The group $SU(2)$ is the set of complex matrices $\begin{pmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix}$ with determinant 1. As a set it coincides with the sphere in $\mathbb{C}^2$: $\{|z_1|^2 + |z_2|^2 = 1\}$, and hence the unit sphere S^3 in $\mathbb{R}^4$, $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$.*

The following is a perhaps better explanation of the group structure on S^3 :

Proposition 1. *The vector space $\mathbb{H}$ of all $h = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, (t, x, y, z) \in \mathbb{R}^4$, where multiplication is defined by relations $\mathbf{i} \times \mathbf{j} \times \mathbf{k} = \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ is an associative algebra, with multiplicative unit $\mathbf{1}$. Defining*

$$\overline{h} = t - (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

we have that $N : \quad h \mapsto h\overline{h} = x^2 + y^2 + z^2 + t^2$ is a morphism ($N(h \times h') = N(h)N(h')$), and any non nil element of $\mathbb{H}$ is hence invertible with $h^{-1} = \dfrac{\overline{h}}{\sqrt{N(h)}}$. Hence $\mathbb{H}$ is a (non abelian) field.[1]

The group of units is $S = \operatorname{Ker} N = \{h \in \mathbb{H} \mid N(h) = 1\}$, it is isomorphic à $SU(2)$.[2]

Remarque 2. *Two reasonable parametrisations of $G = S^3$ are :*

- $z_1 = \cos\theta\, e^{i\varphi}, z_2 = \sin\theta\, e^{i\psi}$ *with* $\theta \in [0, \pi/2], 0 \leqslant \varphi, \psi \leqslant 2\pi$.
- $t = \cos\theta, x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi\cos\theta, z = \sin\theta\sin\varphi\sin\theta$ *(spherical coordinates in $\mathbb{R}^4$ with $r = 1$) where for instance $0 \leqslant \theta, \varphi \leqslant \pi$ and $0 \leqslant \psi \leqslant 2\pi$.*

[1] The quaternions were invented by Hamilton in 1846.

[2] $\mathbb{H}$ is isomorphic to the whole algebra of matrices $\begin{pmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix}$.

Théorème 6. *Any Haar measure on $G = S^3$ is a multiple of $\mu_1 = \sin 2\theta \; d\theta \; d\varphi \; d\psi$ and $\mu_2 = \sin^2\theta \sin\varphi \, d\theta \, d\varphi \, d\psi$.*

Remarque 3.

- *They are invariant, non only by translation, but also by antipody ($h \mapsto -h$) and inversion ($h \mapsto h^{-1} = \overline{h}$), which may have some musical meaning.*
- *We have $\mu_1(G) = 4\pi^2$, but $\mu_2(G) = 2\pi^2$ hence the measures are not equal, but proportional. Invariance by translation is non trivial, but for the other operations it is almost obvious.*

These explicit expressions are derived in [5, 6].

3. **Interval function.** We follow David Lewin in [8], defining the interval function between A, B (measurable subsets of G) by, for any g, the size of the part of A which, when translated by g, falls in B :

Définition 3. *$\mathit{IFunc}_r(A, B): \; g \mapsto \mu(A \cap B \times g^{-1}) = \int_G 1_A(h) 1_B(h \times g) \, d\mu(h)$.*

Here we denote by 1_A the characteristic function of A in order to use a familiar integral representation of μ. Notice that $\mathrm{IFunc}_r \neq \mathrm{IFunc}_l$, where

Définition 4. *$\mathit{IFunc}_l(A, B): \; g \mapsto \mu(A \cap g^{-1} \times B) = \int_G 1_A(h) 1_B(g \times h) \, d\mu(h)$.*

Using both invariances (left and right) of the Haar measure on G, we get

Théorème 7.

(1) $\mathit{IFunc}_r(A, g \times B) = \mathit{IFunc}_r(g^{-1} \times A, B)$
(2) $\forall g, t \in G, \;\; \mathit{IFunc}_r(A, B)(g \times t) = \mathit{IFunc}_r(A \times g, B)(t)$
(3) $\forall g \in G, \;\; \mathit{IFunc}_r(A, B)(g^{-1}) = \mathit{IFunc}_r(B, A)(g)$

and expressions of the same ilk for IFunc_l. All derivations proceed in similar ways :

Preuve. (1)

$$\mathrm{IFunc}_r(A, g \times B)(t) = \int_G 1_A(h) 1_{g \times B}(h \times t) \, \mathrm{d}\mu(h) = \int_G 1_A(h) 1_B(g^{-1} \times h \times t) \, \mathrm{d}\mu(h)$$
$$= \int_G 1_A(g \times h') 1_B(h' \times t) \, \mathrm{d}\mu(g \times h') = \int_G 1_{g^{-1} \times A}(h') 1_B(h' \times t) \, \mathrm{d}\mu(h')$$

using invariance by left translation ($h = g \times h'$) ; hence
$\mathrm{IFunc}_r(A, g \times B) = \mathrm{IFunc}_r(g^{-1} \times A, B)$.

(2)

$$\mathrm{IFunc}_r(A, B)(g \times t) = \int_G 1_A(h) 1_B(h \times g \times t) \, \mathrm{d}\mu(h) = \int_G 1_{A \times g}(h \times g) 1_B(h \times g \times t) \, \mathrm{d}\mu(h \times g)$$

(invariance by right translation here) $= \int_G 1_{A \times g}(h') 1_B(h' \times t) \, \mathrm{d}\mu(h') = \mathrm{IFunc}_r(A \times g, B)(t)$

(3) $\mathrm{IFunc}_r(A, B)(g^{-1}) = \int_G 1_A(h') 1_B(h' \times g) \; \mathrm{d}\mu(h') = \int_G 1_A(h \times g^{-1}) 1_B(h) \; \mathrm{d}\mu(h) = \mathrm{IFunc}_r(B, A)(g)$. Again invariance by right translation, having $h' = h \times g^{-1}$.

□

4. **Non abelian hexachordal theorem.** We denote by $\bar{A}$ the complementaary set $G \setminus A$. As we said above,

Définition 5. *The right interval content of $A \subset G$ is the map*

$$iv_d(A): \; g \mapsto \mu(A \cap (A \times g)) = \int_G 1_A(h) 1_A(h \times g^{-1}) \, d\mu(h)$$

The left interval content is defined similarly. We state a more precise form for the main theorem of this paper :

Théorème 8. *The map $A \mapsto iv(A) - \mu(A)$ is invariant by complementation. When $\mu(A) = \mu(\bar{A})$ ("generalized hexachords"), then $iv(A) = iv(\bar{A})$.*

Notice that $1_A + 1_{\bar{A}} = 1 (= 1_G)$ and $1_A^2 = 1_A$. We generalize the simplest known computation for the classical hexachordal theorem, such as it can be found in [4].[3] Our own favorite proof [1], using Fourier Transform, works without modification for compact abelian groups, but is unsuitable for the non abelian case.

Preuve.

$$iv(\bar{A})(g) = \int_G 1_{\bar{A}}(h)1_{\bar{A}}(h \times g^{-1}) = \int_G (1-1_A)(h)(1-1_A)(h \times g^{-1}) = \int_G 1-1_A(h) - \int_G 1_A(h \times g^{-1}) + \int_G 1_A(h)1_A(h \times g^{-1})$$

Hence, since (by right invariance) $\int_G 1_A(h \times g^{-1}) = \mu(A \times g) = \mu(A)$, we get $iv(\bar{A}) = \mu(\bar{A}) - \mu(A) + iv(A)$, qed. □

Remarque 4. *The total measure of G is irrelevant : any multiple of the unit Haar measure can be used, eg μ_1 and μ_2 are both suitable in the example of $G = S^3$.*

III. Perspectives

It is difficult to envision at present how to generalize further this version of the hexachordal theorem. Of course, it might be the basis for asymptotic methods (quite popular in mathematical research nowadays) and some meaning could be given, perhaps, to $iv(A) - iv(\bar{A})$ even though neither A nor its complement have finite measure. Still, if one wishes to retain a grasp on the intuitive origin of the concept, one cannot dispense with an overall finite measure (so that both a subset and its complement may have one, too) and a Haar measure in order to define interval vectors. Furthermore, as we have seen in the computations, if IFunc is to make sense in any reasonable way, then the Haar measure has to be right-invariant too. Since some of those same computations make use of inverse translations, it is hard to envision less than a topological group structure on the ambiant set. For instance, a Haar measure on a monoid would be a rare bird indeed. "Locally compact" is required in order to ensure the existence of a Haar measure (conveniently unique up to a scalar), but overall compactness is required both for finite measures and for left and right Haar measures being the same. On the other hand, the "separable" hypothesis, while mathematically mandatory, is not a serious drawback in practice.

In a non abelian group, $iv_r \neq iv_l$: for example in the group G of affine transformations in $\mathbb{Z}_{12}$, which is a non abelian group with 48 elements, consider the subset $A = \{x \mapsto 3 + 5x, 5 + 7x, 2 + 7x, 4 + x, 4 + 3x\}$. The Haar measure is the counting measure. Now the maps iv_r and iv_g differ in some values (not many : 8 out of 48). For instance, if $g = (x \mapsto 1 + 3x)$ we get

$$g \circ A = \{2 + 7x, 5x, 7 + 5x, 5 + 3x, 5 + x\} \text{ while } A \circ g = \{7x, 4 + 5x, 1 + 5x, 5 + 3x, 7 + x\}$$

This raises the interesting notion of those subsets $A \subset G$ for which $iv_r = iv_l$. For instance,subgroups are such subsets :

Proposition 2. *Let H be a subgroup of G. Then*

$$iv_l(g) = iv_r(g) = \begin{cases} \mu(H) & \text{when } g \in H \\ 0 & \text{else} \end{cases}.$$

but there are others, like $x \mapsto x, x+1, x+2, 5x+3$ in the affine group above. Suprisingly many of them : for instance in G above, 4,696 of the 35,960 4-elements subsets share this feature. Empirical evidence suggests a ratio above 10% of all subsets in general. For a last example, in the good old T/I diedral group of transpositions and inversions in $\mathbb{Z}_{12}$ there are 2,010 4-elements subsets (out of 10,626) such that $iv_r = iv_l$.

As for other spheres, there is little hope, if at all, of expanding the hexachordal theorem to S^2 or S^n in general. As we discussed, we need a group structure in the set in order to define $A + t$ (or more generally $g \times A$ before we even attempt to compute a measure of $A \cap (g \times A)$.[4] There is one larger sphere where the present result *might* work : S^7 inherits from Cayley's octavions a quasi-group structure - but it is not associative. Since

[3] Though this proof has been around in the US - personal communication form D. Tymoczko - we did not see it in print elsewhere ; for instance in Rahn's textbook *Basic Atonal Theory* the proof is "left as a useful exercise". Useful indeed !

[4] The computations showed well that combination and inversion of elements are necessary, with associavity to boot.

spheres other than S^0, S^1, S^3 and S^7 are not parallelizable they cannot be Lie groups[9] (we do not know if they can still be provided with continuous, but non differentiable, topological group structures) and this leaves very little hope indeed of a hexachordal theorem on them.
On the other hand, a lesser structure may do as well : following a result by J. Mandereau (to be published in [7]), an autocorrelation function iv can be inherited on the quotient of a group with a Haar measure by any of its subgroups, *even if not normal* : the set of cosets G/H (left or right) does not have itself to be a group. Extension of this result to the non abelian case would enrich considerably the field of sets with a (double !) hexachordal theorem. Moreover, considering the particular shapes of iv functions on ME sets, it may be possible at long last to enlarge their concept to a whole new class of objects. We would like to stress that in particular, the generalized hexachordal theorem is valid in all finite groups with a particularly simple Haar measure which is simply the counting measure. Another nice feature is that, though we have to contend with two hexachordal theorems instead of one, they are both true for any reasonably defined way of measuring subsets, since the Haar measure is unique.[5]

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[5] This is true even in locally compact groups.