Can a musical scale have 14 generators?

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Abstract
A finite arithmetic sequence of real numbers has exactly two generators: the sets $\{a, a+f, a+2f, \dots a+(n-1)f = b\}$ *and* $\{b, b-f, b-2f, \dots b-(n-1)f = a\}$ *are identical. A different situation exists when dealing with arithmetic sequences modulo some integer c. The question arises in music theory, where a substantial part of scale theory is devoted to generated scales, i.e. arithmetic sequences modulo the octave. It is easy to construct scales with an arbitrary large number of generators. We prove in this paper that this number must be a totient number, and a complete classification is given.*

In other words, starting from musical scale theory, we answer the mathematical question of how many different arithmetic sequences in a cyclic group share the same support set. Extensions and generalizations to arithmetic sequences of real numbers modulo 1, with rational or irrational generators and infinite sequences (like Pythagorean scales), are also provided.

KEYWORDS: Musical scale, generated scale, generator, interval, interval vector, DFT, discrete fourier transform, arithmetic sequence, music theory, modular arithmetic, cyclic groups, irrational.

Foreword

In the cyclic group $\mathbb{Z}_c$, arithmetic sequences are not ordered as they are in $\mathbb{Z}$. For instance, the sequence (0 7 14 21 28 35 42) reduces modulo 12 to (0 7 2 9 4 11 6), whose support is (rearranged) $\{0, 2, 4, 6, 7, 9, 11\} \subset \mathbb{Z}_{12}$. In music theory, such arithmetic sequences in a cyclic group are the basis of studies on Maximally Even Sets, Well-Formed Scales [5] and other major topics of scale theory,[1], but also of many rhythms, e.g. the Tresillo 0 3 6 mod 8 which is so prominent in Latin America's music.

In the example above, the G major scale is generated by $f = 7$, of backwards by $-f = 5$ using the convention C=0. All twelve diatonic scales can be generated thusly.

It seems natural to infer that, as in $\mathbb{Z}$, such arithmetic sequences have exactly two possible (and opposite) generators. But such is not the case.

1. The 'almost full' scale, with $c-1$ tones, is generated by any interval f coprime with c: such a scale is $f, 2f, \dots (d-1)f$ mod c^2. For $c = 12$ for instance we have $\Phi(12) = 4$ different generators, Φ being Euler's totient function. *Idem* for the full aggregate, i.e. $\mathbb{Z}_c$ itself.

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[1] I am indebted to a reviewer for reminding me of Mazzola's 'circle chords' which provide 'a generative fundament for basic chords in harmony', cf. [12] pp. 514 for a reference in English.

[2] This was mentioned to Norman Carey by Mark Wooldridge, see [4], chap. 3.

2. When f generates a scale which is a subgroup/coset of $\mathbb{Z}_c$, then kf generates the same scale as f, for any k coprime[3] with c, that is to say the generators are all those elements of the group $(\mathbb{Z}_c, +)$ whose order is equal to some particular divisor of c. Consider for instance a 'whole-tone scale' in a 14-tone chromatic universe, i.e. a 7-element sequence generated by 2 modulo 14. It exhibits 6 generators, which are the elements of $\mathbb{Z}_{14}$ with order 7, namely the even numbers: the set $\{0, 2, 4, 6, 8, 10, 12\}$ is produced by either of the 6 following sequences

$$(024681012)(048122610)(061241028)(082104126)(010621284)(012108642)$$

   Mathematically speaking, the subgroup of $\mathbb{Z}_c$ with d elements where d divides c, is generated by precisely $\Phi(d)$ intervals. Conversely, the subgroup of $\mathbb{Z}_c$ generated by f is the (one and only) cyclic subgroup with $c/\gcd(c, f))$ elements.
3. One last example: the 'incomplete whole-tone scale' in 10-tone chromatic universe $\{1, 3, 5, 7\}$ has 4 different generators, namely 2, 4, 6, 8.

The present paper studies all possible arithmetic sequences (a.k.a. generated scales) $\{a, a+f, a+2f, \dots a+(d-1)f\}$ in $\mathbb{Z}_c$. We will prove that the above examples cover all possible cases, hence the number of generators is always a 'totient number' $\Phi(n)$.[4]

- When a generator f is coprime with c, there are two generators only, except in the cases of the full and 'almost full scale' (when $d = c$ or $d = c - 1$) which admit $\Phi(c)$ generators.
- When some generator of the scale is not coprime with c, it will be seen that the number of generators can be arbitrarily large. At this point the whole classification shall be obtained.
- Then we will endeavour to bring the question into a broader focus, considering partial periodicity and its relationship to Discrete Fourier Transform.
- Lastly, these results will be generalized for scales with non-integer generators.

Several subcases and developments had to be removed because of the size limits.

Notations and conventions

- Unless otherwise mentioned, computations take place in $\mathbb{Z}_c$, the cyclic group with c elements whose elements are 'tones'.
- $\mathbb{Z}$ (respectively $\mathbb{N} = \mathbb{Z}^+$) stands for the integers (respectively the non-negative integers).
- $a \mid b$ means that a is a divisor of b in the ring of integers.
- The symbol $\lfloor t \rfloor$ denotes the floor function, i.e. the greatest integer lower than, or equal to t.
- The word 'scale' is used, incorrectly but according to custom, for 'pc-set', i.e. an unordered subset of $\mathbb{Z}_c$. Some authors have provided more correct definitions but I shall use this one since the main point of this paper is the possibility of different sequencings.

[3] This case was suggested by David Clampitt in a private communication; it also appears in [13].
[4] Sloane's integer sequence A000010.

- A 'generated scale' is a subset of $\mathbb{Z}_c$ which can be build from the values of some finite arithmetic sequence (modulo c), e.g. $A = \{a, a+f, a+2f \dots\} \subset \mathbb{Z}_c$.
- 'ME set' stands for Maximally Even Set',[5] 'WF' means 'Well Formed', 'DFT' is 'Discrete Fourier Transform'.
- Φ is Euler's totient function, i.e. $\Phi(n)$ is the number of integers smaller than n and coprime with n.
- We will say that $A \subset \mathbb{Z}_c$ is generated by f if A can be written as $A = \{a, a+f, a+2f \dots\}$ for some suitable starting point $a \in A$. Notice that sets are denoted using curly braces, while sequences are given between parentheses (somewhat salvaging the sloppiness of the definition of a 'scale')

1 Results

The different cases discussed have no one-to-one relationship with the different cases in the conclusion.

1.1 The simpler cases

For any generated scale *where the generator f is coprime with c*, there are only two generators f and $c - f$, except for the extreme cases mentioned in the foreword:

Proposition 1
Let $1 < d < c-1$; the scales $A = \{0, a, 2a, \dots (d-1)a\}$ and $B = \{0, b, 2b, \dots (d-1)b\}$ with d tones, generated in $\mathbb{Z}_c$ by intervals a, b, where one at least of a, b is coprime with c, cannot coincide up to translation, unless $a = b$ or $a + b = c$ (i.e. $b = -a \mod c$). In other words, A admits only the two generators a and $-a$.

But when $d = c - 1$ or $d = c$, then there are $\Phi(c)$ generators.

(all non-trivial proofs at the end of the paper)

Another simple case allows multiple generators:

Proposition 2 *A* regular polygon *in $\mathbb{Z}_c$, i.e. a translate of some subgroup $f\mathbb{Z}_c$ with d elements, has exactly $\Phi(d)$ generators.*

Clearly in that case f will not be coprime with c (except in the extreme case of $d = c$ when the polygon is the whole aggregate $\mathbb{Z}_c$).

1.2 The remaining cases

[5] These sets were introduced by Clough and Myerson [7], they can be seen as scales where the elements are as evenly spaced as possible on a number of given sites and comprise the diatonic, whole-tone, pentatonic and octatonic scales among others. See also [6, 8, 1].

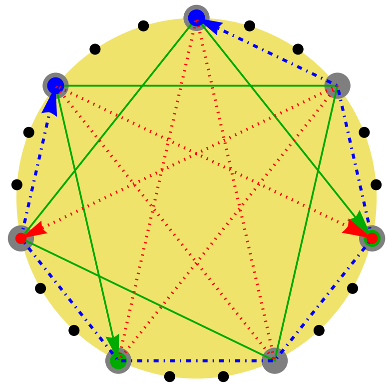

Fig. 1. Many generators for a regular polygon

Theorem 1 *A scale generated by f **not** coprime with c, with a cardinality $1 < d < c$, has*

1. *one generator when the scale is (a translate of) $\{0, c/2\}$ (a 'tritone');*
2. *two generators (not coprime with c) when d is strictly between 1 and $c' - 1 = c/m - 1$ where $m = \gcd(c, f)$;*
3. *$\Phi(d)$ generators when $d = c' = c/m$, i.e. when A is a regular polygon;*
4. *$\Phi(d+1)$ generators when $d = c' - 1$, and A is a regular polygon minus one point; all generators share the same order in the group $(\mathbb{Z}_c, +)$;*

The third case is actually that of Prop. 1.

The last, new case, features *incomplete regular polygons*, i.e. regular polygons with one point removed. For any divisor c' of c, let $f = c/c'$ and $d = c' - 1$:

$$\{f, 2f, \dots d\,f \pmod{c}\} \qquad \text{where } d\,f = c - f = -f$$

is the simplest representation of such a scale.[6]

Consideration of the 'gap' in such scales yields an amusing fact, whose proof will be left to the reader:

Proposition 3 *For each different generator of an 'incomplete polygon', there is a different starting point of the arithmetic sequence.*

Hence

Theorem 2 *There are three cases of generated scales: regular polygons, regular polygons minus one tone, and 'diatonic-like' scales, i.e. affine images of a chromatic segment, as summarized in picture 2. The number of generators of such a scale can be any totient number.*

[6] A reviewer sums up nicely the two 'plethoric' cases by identifying them with multiple orbits of affine endomorphisms, i.e. different affine maps generating the same orbit-sets. See also [2] about orbits of affine maps modulo n.

Conversely, *nontotient numbers*, that is to say numbers that are not a $\Phi(n)$, can never be the number of generators of a scale. All odd numbers (apart from 1) are nontotient, the sequence of even nontotients begins with 14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98…[7]

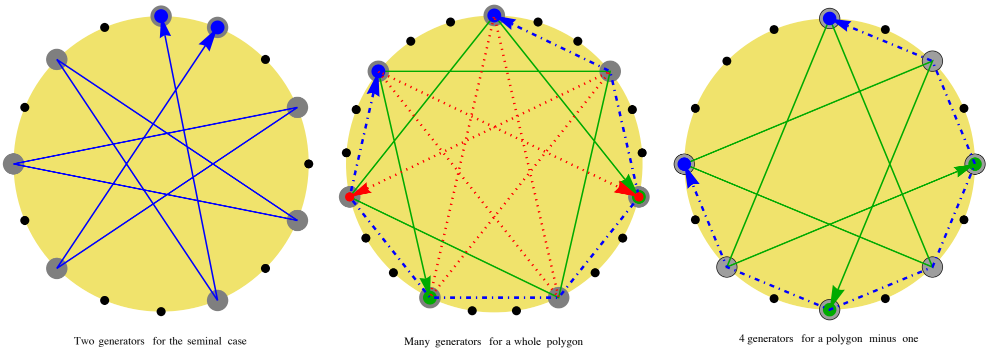

Fig. 2. Three cases

1.3 Partial periodicities and Fourier Transform

In the seminal case, the values of the two generators can be conveniently recognized as the indexes of the maximum Fourier coefficients of the scale. Informally, a generated scale A is at least partially periodic, and this is apparent on the Fourier transform $\mathcal{F}_A : t \mapsto \sum_{k\in A} e^{-2i\pi\, k\, t/c}$.

More precisely, in the online supplementary of [1] it is established that

Theorem 3 *For c, d coprime, a scale with d tones is generated by an interval f, coprime with c, if and only if the semi-norm*

$$\|\mathcal{F}_A\|^* = \max_{t \text{ coprime with } c} |\mathcal{F}_A(t)| = \max_{t \text{ coprime with } c} \Big|\sum_{k\in A} e^{-2i\pi\, k\, t/c}\Big|$$

is maximum among all d-element scales, i.e. for any scale B with d elements, $\|\mathcal{F}_A\|^* \geq \|\mathcal{F}_B\|^*$. Moreover, if $\|\mathcal{F}_A\|^* = |\mathcal{F}_A(t_0)|$, then $t_0^{-1} \in \mathbb{Z}_c$ is one generator of scale A, the only other being $-t_0^{-1}$.

This extends the discovery made by Ian Quinn [13]:

Quinn's Fourier Characterization 1 *A is Maximally Even with d elements if, and only if, $|\mathcal{F}_A(d)|$ has maximum value among d-subsets.*

[7] Sloane's sequence A005277 in his online encyclopedia of integer sequences [15]. For the whole sequence including odd numbers, see A007617.

The connexion being that the scales involved in Thm. 3 are affinely equivalent to some ME sets, which are themselves affine images of chromatic clusters (see also proof of Prop. 1). Informally it vindicates Quinn's idea of *saliency* or chord quality: a pc-set with a large magnitude of its k^{th} coefficient is c/k-ish (or k^{-1}-ish in $\mathbb{Z}_c$ as the case may be), e.g. (for $c = 12$) a large third coefficient means *major-thirdish*, a large fifth coefficient means *fifthish*.

1.4 Non-integer generation

The present paper covered exhaustively the case of scales of the form

$$\{a + k\,f \pmod c, k = 0 \dots d-1\}, a, c, f \in \mathbb{Z}$$

but in other music-theoretic models, generation from non-integer steps is also common. There are two important cases: for a non-integer or even irrational generator f one can still retrieve integers modulo some c by rounding up, which gives rise to Clough and Douthett's J-functions:

$$J_\alpha(k) = \lfloor k\alpha \rfloor \pmod c, \quad \text{wherein usually } \alpha = \frac{c}{d}$$

Let us call J_α-sets the sets of the form $\{J_\alpha(0); \dots J_\alpha(d-1)\} \subset \mathbb{Z}_c$.[8] When $\alpha = c/d$, the J_α-set is a Maximally Even set.

The second important case adresses finite or even infinite arithmetic sequences in the continuous circle, with the seminal notion of $\mathcal{P}_x$-sets, made up with consecutive values of the maps $k \mapsto \mathcal{P}_x(k) = k\,x \pmod 1$, that express pythagorean-style scales (e.g. $x = \log_2(3/2)$). Some of them are the Well-Formed Scales [4, 5]. The question of different generators can be formulated thus:

If two sets of values of J_α, J_β (resp. $\mathcal{P}_x$, $\mathcal{P}_y$) are transpositionally equivalent, do we have necessarily $\alpha = \pm\beta$ (resp. $x = \pm y$)? In other words, do such scales have exactly 2 generators and no more?

We have already seen counter-examples, for instance in Prop. 2 when $\alpha \in \mathbb{N}, d\,\alpha = c$ and the J_α-set is a regular polygon (or similarly $d\,x \in \mathbb{N}$ for the $\mathcal{P}_x$ case). Other cases are worth investigating.

Let us first consider the values of a J function with a random multiplier, e.g. $J_\alpha(k) = \lfloor k\,\alpha \rfloor \pmod c$ with some $\alpha \in \mathbb{R}$. These values have been mostly scrutinized when $\alpha = c/d$. Since the floor function is locally right-constant, $J_\alpha(k)$ does not change for a small increase of α and hence

Proposition 4 *A J_α-set (up to translation) does not characterize the pair $\pm\alpha$: there are infinitely many real α's that generate the same sequence (the set of those α's has strictly positive measure).*

Secondly, we state a result when the generator x, in a finite sequence of values of $\mathcal{P}_x$, is irrational:

[8] Some degree of generalization is possible, see [6] for instance, but results merely in translations of the set.

Theorem 4 *If the sets* $\mathcal{P}_x^d = \{0, x, 2x, \dots (d-1)x\} \pmod 1$ *and* $\mathcal{P}_y^d = \{0, y, 2y, \dots (d-1)y\} \pmod 1$ *are transpositionally equivalent, with* x *irrational and* $d > 0$*, then* $x = \pm y \pmod 1$.

This has been implicitely known in the case of Well-Formed Scales in non-tempered universes [4], but this theorem is more general.

Lastly, we characterize the generators of the 'infinite scales'[9] $\mathcal{P}_x^\infty = \{nx \pmod 1, n \in \mathbb{Z}\}$:

Theorem 5 *Two infinite generated scales are equal up to translation, i.e.* $\exists \tau, \mathcal{P}_x^\infty = \tau + \mathcal{P}_y^\infty$*, if and only if they have the same generator up to a sign, i.e.* $x = \pm y \pmod 1$*, when* x *is irrational.*

In the case where x *is rational and* $x = a/b, \gcd(a, b) = 1$*, there are* $\Phi(b)$ *different possible generators (just as in the finite case, see Thm. 2).*

In other words, an infinite Pythagorean scale has 2 or $\Phi(b)$ generators, according to whether the number of actually different tones is infinite or finite. The special (tritone) case of one generator already mentioned in Prop. 1 also occurs for $x = 1/2$.

2 Proofs

2.1 Proof of Prop. 1

The following proof relies on the one crucial concept of (oriented) interval vector,[10] that is to say the multiplicities of all intervals inside a given scale. This is best seen by transforming A, B into segments of the chromatic scale, by way of affine transformations.

Proof. All computations are to be understood modulo c. The extreme cases $d = c - 1$ and $d = c$ were discussed before: any f coprime with c generates the whole $\mathbb{Z}_c$, hence $\{f, 2f, \dots (d-1)f\}$ is always equal to $\mathbb{Z}_c$ deprived of 0. Up to a change of starting point, we can generate with such an f any subset with cardinality $d - 1$.

Furthermore, a generator f **not** coprime with c would only generate (starting with 0, without loss of generality) a part of the strict subgroup $f\mathbb{Z}_c \subset \mathbb{Z}_c$, hence a subset with stricly less than $c - 1$ elements.

So we are left with the general case, $1 < d < c - 1$. Without loss of generality we take b invertible modulo c (a and b are interchangeable, and we assumed that one of them is invertible modulo c, i.e. coprime with c).

Let $D = \{0, 1, 2, \dots d - 1\}$; assume $A = B + \tau$, as $A = aD$ and $B = bD$, then D must be its own image ($\varphi(D) = D$) under the following affine map:

$$\varphi : x \mapsto b^{-1}(a\,x - \tau) = b^{-1}ax - b^{-1}\tau = \lambda x + \mu.$$

[9] I have to stress the musical interest of such bizarre objects, actively researched both in the domain of word/scale theory [4, 5, 9] and aperiodic rhythms [3] and providing compositional material.

[10] A famous concept in music theory, see for instance [14]. I had initially found an alternative proof based on majorizations of the Fourier Transform, omitted here in favour of a shorter one.

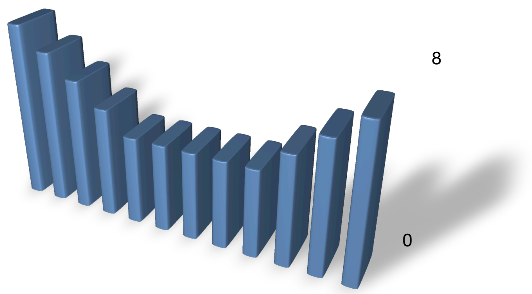

Fig. 3. Interval vector of (0 1 2 3 4 5 6 7) mod 12.

We now elucidate the different possible multiplicities of intervals between two elements of D.

Lemma 1 *Let* $1 < d < c-1$*, we define the oriented interval vector* $\boldsymbol{IV}_D$ *by* $\boldsymbol{IV}_D(k) = \#\{(x,y) \in D^2, y-x = k\}$*. Then* $\boldsymbol{IV}_D(k) < d-1\ \forall k = 2 \dots c-2$*; to be more precise,*

$$\boldsymbol{IV}_D = [\boldsymbol{IV}_D(0), \boldsymbol{IV}_D(1), \boldsymbol{IV}_D(2), \dots \boldsymbol{IV}_D(c-1)] = [d, d-1, d-2, d-3 \dots, d-3, d-2, d-1]$$

i.e. interval 1 and its opposite $c-1$ *are the only ones with multiplicity* $d-1$*.*

For instance, with $c = 12, d = 8$, one computes $\mathbf{IV}_D = [8, 7, 6, 5, 4, 4, 4, 4, 4, 5, 6, 7]$.[11] See a picture of this interval vector on fig. 3.

Proof. This can proved by induction, or by direct enumeration. For want of space we leave it as an exercise. Fig. 4 may help to distinguish the two different ways of getting interval k, either between consecutive elements (dashed) or across the gap (dotted).

As we will see independently below, if one generator is invertible, then so are all others.[12] Hence $\lambda = b^{-1}a$ is invertible in $\mathbb{Z}_c$ and the map $\varphi : x \mapsto \lambda x + \mu$ above is one to one; it multiplies all intervals by $\lambda \mod c$:

$$\varphi(j) - \varphi(i) = (\lambda j + \mu) - (\lambda i + \mu) = \lambda.(j - i),$$

which turns the interval vector $\mathbf{IV}_D$ into $\mathbf{IV}_{\varphi(D)}$ wherein $\mathbf{IV}_D(\lambda i) = V_{\varphi(D)}(i)$: the same multiplicities occur, but for different intervals.[13]

Most notably, the *only*[14] two intervals with multiplicity $d-1$ in $\mathbf{IV}_{\varphi(D)}$ are λ and $-\lambda$. Hence, if $\mathbf{IV}_{\varphi(D)} = \mathbf{IV}_D$, the maximal multiplicity $d-1$ must appear in positions 1 and $c-1$, which compels λ to be equal to ± 1. Finally, as $\lambda = a\,b^{-1} \mod c$, we have indeed proved that $a = \pm b$, qed.

[11] The minimum value of **IV** and the number of its repeated occurrences could be computed – it is 0 for $d < c/2$ – but are irrelevant to the discussion.

[12] This could also be proved directly from $\varphi(D) = D$.

[13] It is well known that affine transformations permute interval vectors.

[14] Because φ is one to one.

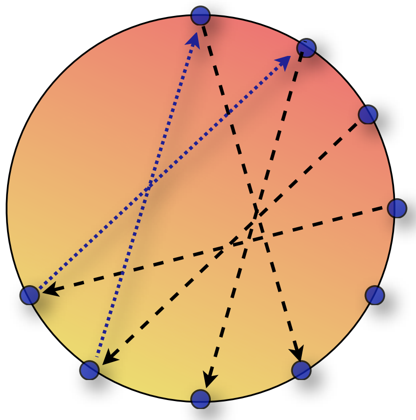

Fig. 4. Double origin of one interval.

2.2 Proof of Prop. 2

A regular polygon is a translate of a subgroup with d elements, and such a subgroup has $\Phi(d)$ generators.

2.3 Proof of Thm. 1

The first case is obvious, when c is even the only generator of $\{0, f = c/2\}$ is $c/2$. Case 3 was studied in Prop. 2.

There remains to be considered the case of a scale generated by some f **not** coprime with c, when that scale is not a regular polygon. For the end of this discussion, let $\gcd(f, c) = m > 1$, and assume $d > 1$ and (up to translation) $0 \in A$.

Lemma 2 *If f, g are two generators of a same scale A, then $m = \gcd(c, f) = \gcd(c, g)$.*

Proof. Consider the group D_A of differences of elements of A, generated by iteration (here a finite number thereof) of the operator $\Delta : A \mapsto A - A$. This is a subgroup of $\mathbb{Z}_c$, hence a cyclic group, i.e. some $m\mathbb{Z}_c$.[15] Now for any generator f of A, it is obvious that D_A is also generated by f (any difference of elements of A being a multiple of f). The order of f is the cardinality c/m of D_A, hence the Lemma.
NB: this lemma was initially proved using DFT.

As a special case, this proves what we had advanced during the proof of Prop. 2, namely that if one of the generators of a scale is invertible modulo c, then so are all others.

From there, one can divide A by m and assume without loss of generality that $f' = f/m$ and $c' = c/m$ coprime. We are dealing now with a scale $A' = A/m$ in

[15] This is the group generated by $A - A$, in all generality, cf. [12], 7.26.

$\mathbb{Z}_{c'}$, generated by f' and $g' = g/m$, both coprime with c': then Prop. 1 provides two cases, either $\#A' = d < c' - 1$ or not. In the latter case, we have $\Phi(c')$ generators for an almost full, or full, regular polygon; in the former, only two, like for the generic 'diatonic-like' scale.

As for instance, $f' = \pm g' \pmod{c)'} \iff f = \pm g \pmod c$, we have exhausted all possible cases when a generator is not coprime with c, and proved Prop. 1, hence Thm. 2.

2.4 Proof of Thm. 4

Proof. It is the same the idea as the proof of Prop. 1, using the interval vector; but since the universe is so different (infinite and continuous) I feel the need for an independent proof.

This works in both cases because the affine map $\mathcal{P}_x$ is one to one again: since x is irrational, we have the immediate

Lemma 3 $\forall a, b \in \mathbb{Z},\ a\,x \equiv b\,x \pmod 1 \iff a = b.$

(else x would be equal to some $\dfrac{k}{b-a}$).

Consider now all possible intervals in $\mathcal{P}_x^d$, i.e. the $(i-j)x \pmod 1$ with $0 \le i, j < d$. By our hypothesis, these intervals occur with the same multiplicity in $\mathcal{P}_x^d$ and $\mathcal{P}_y^d$. Let us have a closer look at these intervals (computed modulo 1), noticing first that

- There are d different intervals from 0 to $k\,x$, with k running from 0 to $d-1$. They are distinct because x is irrational, see Lemma 3. Their set is $\mathcal{I}_0 = (0, x, 2x \ldots (d-1)x)$.
- From x to $x, 2x, 3x, \ldots (d-1)x$ *and 0*, there are $d-1$ intervals common with I_0, and a new one, $0 - x = -x$. It is new because x is still irrational. For the record, their set is $\mathcal{I}_x = (0, x, 2x \ldots (d-2)x, \boldsymbol{-x})$.
- From $2x$ to the others, $d-1$ intervals are common with $\mathcal{I}_x$ and only $d-2$ are common with the $\mathcal{I}_0$.
- Similarly for $3x, 4x \ldots$ until
- Finally, we compute the intervals from $(d-1)x$ to $0, x, \ldots, (d-2)x, (d-1)x$. One gets $\mathcal{I}_{(d-1)x} = (0, -x, -2x \cdots -(d-1)x)$.

The table fig. 5, not unrelated to fig. 3, will make clear the values and coincidences of the different possible intervals.

So only two intervals (barring 0) occur $d-1$ times in $\mathcal{P}_x^d$ (resp. $\mathcal{P}_y^d$), namely x and $-x$. Hence $x = \pm y$, qed.

2.5 Proof of Thm. 5

Notice that $\mathcal{P}_x^\infty = \{kx \pmod 1, k \in \mathbb{Z}\}$ is a subgroup of the circle (or one-dimensional torus) $\mathbb{R}/\mathbb{Z}$, quotient group of the subgroup of $\mathbb{R}$ generated by 1 and x. All computations are to be understood modulo 1. If we consider a translated version $\mathcal{S} = a + \mathcal{P}_x^\infty$, then the group can be retrieved by a simple difference:

$$\mathcal{P}_x^\infty = \mathcal{S} - \mathcal{S} = \{s - s', (s, s') \in \mathcal{S}^2\}$$

Intervals starting in										
0					0	x	2x	…	*(d-2)x*	*(d-1)x*
x				-x	0	x	2x	…	*(d-2)x*	
2x			-2x	-x	0	x	2x	…		
…										
(d-2)x		-*(d-2)x*	…	-x	0	x				
(d-1)x	-*(d-1)x*	…	…	-x	0					

Fig. 5. The different intervals from each starting point

So the statement of the theorem can be simplified, without loss of generality, as "if $\mathcal{P}_x^\infty = \mathcal{P}_y^\infty$ then $x = \pm y \pmod 1$" (and similarly in the finite case).

Proof. We must distinguish the two cases, whether x is rational or not.

- The case x rational is characterized by the finitude of the scale. Namely, when $x = a/b$ with a, b coprime integers (we will assume $b > 0$), then $\mathcal{P}_x^\infty$ is the group generated by $1/b$: one inclusion is clear, the other one stems from Bezout relation: there exists some combination $au + bv = 1$ with u, v integers, and hence

$$1/b = u\,a/b + v = u\,a/b \pmod 1 = \underbrace{a/b + \ldots a/b}_{u \text{ times}}$$

  is an element of $\mathcal{P}_x^\infty$. As $1/b \in \mathcal{P}_x^\infty$, it contains the subgroup generated by $1/b$, and finally these two subgroups are equal.
  The subgroup $< 1/b > \pmod 1$ is cyclic with b elements, hence it has $\Phi(b)$ generators, which concludes this case of the theorem.[16]
- Now assume x irrational and $\mathcal{P}_x^\infty = \mathcal{P}_y^\infty$. An element of $\mathcal{P}_x^\infty$ can be written as $ax \pmod 1$, with $a \in \mathbb{Z}$. Since $y \in \mathcal{P}_x^\infty$ then $y = ax \pmod 1$ for some a. Similarly, $x = by$ for some b. Hence

$$x = abx \pmod 1 \quad \text{that is to say in } \mathbb{Z}, \quad x = abx + c$$

  where a, b, c are integers. This is where we use the irrationality of x: $(1 - ab)x = c$ implies that $ab = 1$ and $c = 0$.
  Hence $a = \pm 1$, i.e. $x = \pm y \pmod 1$, which proves the last case of the theorem.

Remark 1 *The argument about retrieving the group from the (possibly translated) scale applies also if the scale is just a semi-group, e.g.* $\mathcal{P}_x^{+\infty} = \{kx \pmod 1, k \in \mathbb{Z}, k \geq 0\}$ *(or even to many subsets of* $\mathcal{P}_x^\infty$*),which are perhaps a less wild generalization of the usual Pythagorean scale. So the theorem still holds for the half-infinite scales.*

[16] The different generators are the k/b where $0 < k < b$ is coprime with b.

Conclusion

Apart from the seminal case of diatonic-like generated scales, it appears that many scales can be generated in more than two ways. This is also true for more complicated modes of 'generation'.

Other simple sequences appear to share many different generation modes. It is true for instance of geometric sequences, like the *powers* of 3, 11, 19 or 27 modulo 32 which generate the same 8-note scale in $\mathbb{Z}_{32}$, namely $\{1, 3, 9, 11, 17, 19, 25, 27\}$ – geometric progressions being interestingly dissimilar in that respect from arithmetic progressions.[17]

I hope the above discussion will shed some light on the mechanics of scale construction. I thank David Clampitt for fruitful discussions on the subject, and Ian Quinn whose ground-breaking work edged me on to explore the subject in depth, and my astute and very helpful anonymous reviewers.

References

1. Amiot, E., nov. 2007, *David Lewin and Maximally Even Sets*, in: *Journal of Mathematics and Music*, Taylor and Francis **1**(3):152-172.
2. Amiot, E., nov. 2008, *Self Similar Melodies*, in: *Journal of Mathematics and Music*, Taylor and Francis **2**(3):157-180.
3. Callender, C., 2013, *Sturmian canons*, in Proceedings MCM 2013, Montreal, Springer:64-75.
4. Carey, N., 1998, *Distribution Modulo 1 and Musical Scales*, PhD thesis, University of Rochester. Available online.
5. Carey, N., Clampitt, D., 1989, *Aspects of Well Formed Scales,* Music Theory Spectrum, **11**(2),187-206.
6. Clough, J., Douthett, J., 1991, *Maximally Even Sets,* Journal of Music Theory, **35:**93-173.
7. Clough, J., Myerson, G., 1985, *Variety and Multiplicity in Diatonic Systems,* Journal of Music Theory, **29:**249-70.
8. Douthett, J., Krantz, R., 2007, *Maximally even sets and configurations:* common threads in mathematics, physics, and music, Journal of Combinatorial Optimization, Springer. Online: `http://www.springerlink.com/content/g1228n7t44570442`
9. Carey, N., 2013, *Lambda Words: A Class of Rich Words Defined Over an Infinite Alphabet*, Journal of Integer Sequences 01/2013, 16(3):13.3.4
10. Lewin, D., 1959, *Re: Intervalic Relations between two collections of notes*, in: *Journal of Music Theory*, 3:298-301.
11. Lewin, D., 1987, *Generalized Musical Intervals and Transformations*, New Haven:Yale University Press.
12. Mazzola, G., et alii, 2002, *Topos of Music*, Birkhäuser.
13. Quinn, I., 2004, *A Unified Theory of Chord Quality in chromatic universes*, Ph.D. dissertation, Eastman School of Music.
14. Rahn, D., 1980, *Basic Atonal Theory*, Longman, New York.
15. Sloane, N.J.A., *Online Encyclopedia of Integer Sequences*, `https://oeis.org/`

[17] Such geometric sequences occur in Auto-Similar Melodies [2], like the famous initial motive in Beethoven's Fifth Symphony, autosimilar under ratio 3.