

STRUCTURES, ALGORITHMS AND ALGEBRAIC TOOLS FOR RHYTHMIC CANONS

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ABSTRACT

Rhythmic canons were introduced to the musical world with D.T. Vuza's seminal papers in PNM twenty years ago [40]. Since the original exposition, this important notion has known many developments and generalizations. The present self-contained paper is intended both as a reference on all presently known theoretical results, and a catalog of the different methods currently in use for the production of diverse rhythmic canons, whether for compositional or for theoretical purposes.

1. WHAT IS A RHYTHMIC CANON?

The ambitious purpose of this paper is to fill in the gap between theory and practice of rhythmic canons. There is indeed quite a distance between musical canons, even rather intellectual ones like Bach's in the *Goldberg Variations* or *the Art of Fugue*, and Vuza canons such as they are used by some modern composers.

The basic idea of a canon is that some recognizable pattern is repeated with different offsets (usually with different instruments, or at least different voices). Sometimes this pattern (henceforth called *the motif*) is modified (say augmented or retrograded). For rhythmic canons, we need only consider the occurrences of the musical events (notes, for instance), regardless of pitch or timbre or dynamics. Hence, with a motif A , transformations $\tau_i, i \in I$ (that may be only offsetings, i.e. translation in time), a canon will be the reunion $\mathcal{C}$ of the transforms $\cup_{i \in I} \tau_i(A)$.

Say for simplicity's sake that the fundamental beats are modelled by integers, e.g. some subset $D \subset \mathbb{Z}$. If a motif is identified with its characteristic function $1_A : D \rightarrow \{0, 1\}$, then the superposition of all its copies appears as a sum: $\mathcal{C} = \sum_i 1_{\tau_i(A)}$. For instance, in the common case when all transformations are just different offsets in time, i.e. $\tau_i = T_i = (t \mapsto t + i)$, we get $\sum_i 1_{A+i}$, which is in general the characteristic function of a multiset not a set.

Example 1. Let M be the tango or habanera rhythm $\{0, 3, 4, 6\}$. An infinite canon can be made by offsetting M by -2 and 0 and repeating the sequence with period 8 :



FIGURE 1. A canon with a tango motif

Notice that $\mathcal{C}(4) = \mathcal{C}(6) = 2$ while $\mathcal{C}(5) = 0$, for instance. A neater canon can be made, without gaps or coincidences, by using also retrogradation:



FIGURE 2. A mosaic, with the tango motif M and its retrogradation R

Obviously, if one chooses to consider as possible beats all possible quavers or semi quavers (say) in a definite time-span, then most musical canons in the classical sense will feature from nil to several notes on each beat. We can try to specialize more (theoretically) interesting kinds of canons: we will call them (following [43]) *coverings* and *packings*. A *covering* is a canon where every available beat features at least one note, maybe more. One gets a trivial covering by starting a new copy of the motif on every beat. In other words, with the above notation $\forall t \in D, \mathcal{C}(t) \geq 1$.

A *packing*, conversely, is a canon where there is never more than one note on every possible beat. With the above notation, $\forall t \in D, \mathcal{C}(t) \leq 1$. A trivial packing is made of only one (or even nil!) copy of the motif.

Covering is not only trivial, but lumpy (several notes on the same beat); packing conversely leaves many gaps.

From a mathematical point of view, the obvious way to get a well-defined and interesting problem¹ is to demand one and only note per beat, like in example 2. This is called a *tiling*, i.e. a mosaic with copies of one motif, maybe allowing some deformations (retrogradation, augmentation among other possibilities, apart from translation in time). Hence

$$\text{Mosaics/tilings} = \text{coverings} \cap \text{packings}: \forall t \in D, \mathcal{C}(t) = 1.$$

Most studies (especially on the pure mathematics side) have been devoted to the simplest case of tiling with just one tile and some of its translates, i.e. *mosaic rhythmic canons by translation*. In the one dimensional case (filling every beat with one and only one note) it is equivalent to the problem of tiling $\mathbb{Z}$, see [28, 40]; if the tile is finite, it is equivalent to the tiling of a cyclic group $\mathbb{Z}_n$, as we will develop *infra* in Thm. 2. See already a simple example on figure 3:



FIGURE 3. A mosaic rhythmic canon designed by George Bloch for a greeting card.

This case, the simplest, is already quite formidable: a number of likely conjectures have been disproved, and several open problems remain, among which finding some (easily computable) sufficient and necessary condition for a motif to tile. This case will make the bulk of this paper. On the other hand, there are very few results about other kinds of rhythmic canons:

- (1) Wild's 'trichord theorem' about tiling with three-note motifs and their retrogrades.
- (2) My result about 'tiling modulo p ', a special case of covering with translates of a motif.
- (3) Other coverings with some cultural relevance in central african cultures are asymmetric rhythms, studied by [20], with enumeration results about canons with inner periods (non Vuza canons, cf. *infra*).
- (4) Hajos/deBruijn's 1950 theorem about periodic tilings can be generalized to tilings with any finite number of motifs.
- (5) A necessary condition can be given for a very specific species of tiling by augmentation [21, 34, 3] by way of Galois theory on finite fields.
- (6) In a limited number of cases, autosimilar melodies [2] enable to devise tilings by augmentation, even in several dimensions (see section 3.3) by affine transforms.

¹See however [10] for coverings in nzakara harp canons, and [20] for an unexpected occurrence of non Vuza canons. Another promising line is to impose a given number of notes on each beat, or a condition on this number. See for instance 4.2 below, or [32].

- (7) Sundry efforts in programming yield practical results [21, 34, 42, 13], but without significant theoretical import.

It must be pointed out, however, that in the domain of rhythmic canons, the interplay between computation and theory is twofold: not only does the theory help provide better programs, but computational efforts conversely allow progress on difficult mathematical problems, see among many other examples [5, 25], the algorithm section below, and Andreatta's article in this issue of *PNM* [?].

2. TILINGS OF THE LINE

In this section, we will study the mosaic kind of rhythmic canons, which are modeled as tilings of the integers. If $A \subset \mathbb{Z}$ is the motif, and B is the set of all offsets, we wish for the translates $A + b, b \in B$ to partition the integers, i.e. for the map

$$\begin{aligned} +_{A,B} : A \times B &\rightarrow \mathbb{Z} \\ (a, b) &\mapsto a + b \end{aligned}$$

to be bijective, which can be written as

$$\mathbb{Z} = A \oplus B$$

2.1. Theory.

2.1.1. *From tilings of $\mathbb{R}$ to tilings of $\mathbb{Z}$ to tilings of $\mathbb{Z}_n$.* It must be mentioned briefly that ‘real’ tilings, i.e. tilings of the real line, by translation of a single (bounded) motif, can be essentially reduced to tilings of the integers (see [28, 40]). Conversely, any tiling of the integers $\mathbb{Z} = A \oplus B$ can be turned into a tiling of the real line $\mathbb{R} = (A + [0, 1]) \oplus B$.

Another general result means that we should focus on tilings with a finite motif, since otherwise the tiling may literally contain anything:

Theorem 1. ([33]) *Any direct sum of two finite sets (of integers) can be extended to an infinite direct sum decomposition of $\mathbb{Z}$: If $A \subset \mathbb{Z}$, $B \subset \mathbb{Z}$ are finite and $A + B = A \oplus B$,² then there exist oversets $A' \supset A$, $B' \supset B$ with $\mathbb{Z} = A' \oplus B'$.*

Now the main basic result introduces a period for any tiling with a finite motif. It goes back to Hajos and deBruijn, whose original proofs are perhaps not up to modern standards (see instead [33, 29, 15, 28]).

Theorem 2. *For any tiling of the integers $\mathbb{Z} = A \oplus B$ by translation of the finite motif A , there exists some period $n \geq 1$ for the offsets B , i.e. $B = C \oplus (n\mathbb{Z})$ for some finite set $C \subset \mathbb{Z}$.*

This essential result links tilings with combinatorics, since all significant factors are now finite.

As pointed out by [28], Ex. 1, the result no longer stands if other transformations are allowed: totally non periodic tilings can be constructed from a motif and its retrograde. We can see on the picture below how the range $\llbracket 0, 8 \rrbracket$ can be tiled in two ways by $\{0, 1, 5\}$ and its retrograde, and these two new tiles can be used in turn, to randomly (and non periodically) tile the whole line (fig. 4).

However, the gist of the proof can be preserved³ to get a more general though weaker statement:⁴

Theorem 3. *For any tiling of the integers by translation of a finite number of finite motifs $A_1, \dots, A_r$, there exists a periodic tiling with the same motifs.*

²Meaning that if $a + b = a' + b'$ with $a, a' \in A, b, b' \in B$, then $a = a'$ and $b = b'$.

³The idea is to consider the tiling as a work in progress: at a given step, i.e. when all integers up to some point are covered, there is only a finite number of possible configurations – up to translation of course; by the pigeon-hole principle, at least one configuration must occur more than once, which enables to construct a periodic tiling by repeating the sequence between two such occurrences.

⁴An even more general form appears in [28], Thm. 5.

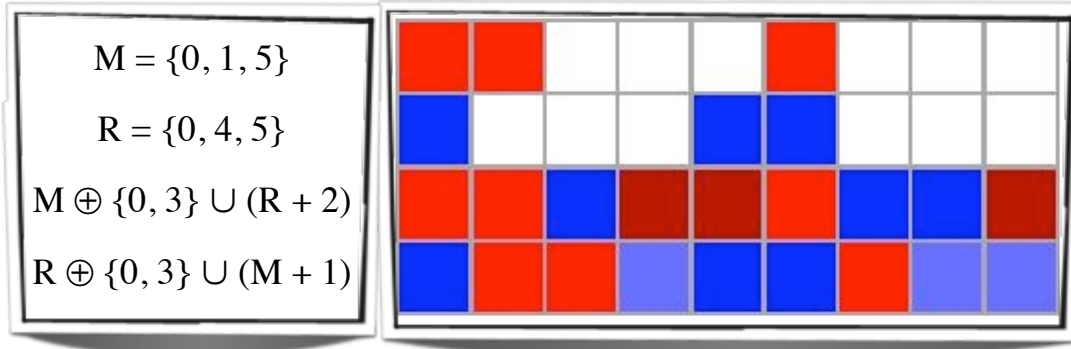
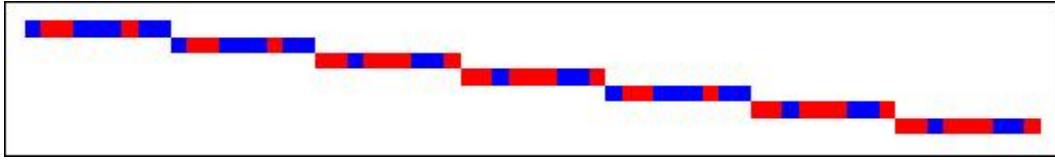
FIGURE 4. Two tilings with retrogradation of $[0, 8]$ with $\{0, 1, 5\}$ 

FIGURE 5. Aperiodic tiling with retrogradation

Starting from equation $\mathbb{Z} = A \oplus (C \oplus n\mathbb{Z})$, we get by reduction modulo n

$$\mathbb{Z}_n = \overline{A} \oplus \overline{C}$$

where $\overline{A}, \overline{C}$ are the sets of residues of the elements of A and C in the cyclic group with n elements $\mathbb{Z}_n$. Moreover, the projection from $\mathbb{Z}$ to $\mathbb{Z}_n$ induces bijections $A \mapsto \overline{A}, C \mapsto \overline{C}$ since its kernel is $n\mathbb{Z}$. So in essence, *tiling the line is the same as finding a decomposition of the generic cyclic group $\mathbb{Z}_n$ as a direct sum of two subsets*. Conversely, given some decomposition $\mathbb{Z}_n = \overline{A} \oplus \overline{C}$, a rhythmic canon emerges by choosing arbitrary integers whose classes modulo n constitute $\overline{A}, \overline{C}$. Consider again example 3: it is a tiling of $\mathbb{Z}$ by $\{0, 4, 5, 9\}$ with offsets $16\mathbb{Z} \oplus \{0, 6, 8, 14\}$, which reduces modulo 16 to $\mathbb{Z}_{16} = \{0, 4, 5, 9\} \oplus \{0, 6, 8, 14\}$.⁵ Other canons would yield the same projection, say with motif $\{4, 9, 16, 21\}$ and offsets $\{0, 8, 22, 30\} \oplus 16\mathbb{Z}$.

Clearly, musically this is a very different canon from Bloch's. Equally clearly, both can be trivially deduced from their common projection on $\mathbb{Z}_{16}$. Henceforth we will study the equation in the cyclic group, written for simplicity

$$\mathbb{Z}_n = A \oplus B \quad (E)$$

The motif A is also called the *inner voice*, and the set of offsets B is the *outer voice*. This last reduction is sometimes considered too drastic by some musicians; it is mandatory, however, if one is to classify and construct canons with a given period.

The first conjecture about the decomposition problem in equation (E) was formulated by Hajos around 1948: he thought that one or the other factor had to be periodic, meaning $\exists p, 0 < p < n, A = A + p$ (or the same with B). An equivalent formulation, reminiscent of Thm. 2, is $A = A' \oplus p\mathbb{Z}_n$, meaning that A is generated by a submotif A' , translated by p and all its multiples (in particular, p must be a strict divisor of n).

Example 2. In the above tiling of $\mathbb{Z}_{16}$ used by G. Bloch, the second factor $\{0, 6, 8, 14\}$ has period 8 and can be written $\{0, 6\} \oplus \{0, 8\}$.

In his seminal paper [40], D.T. Vuza begins by proving this conjecture for $n = 12$, which is of course a vital case for musicians – if classes modulo 12 model pitch-classes instead of beats, then this means that in any Boulezian *multiplication of chords* that yields a tiling of the chromatic aggregate, one of the

⁵We denote identically an integer and its class modulo n , the context usually making clear which is which.

chords is a limited transposition mode in Messiaen's (generalized) sense. Vuza eventually managed all by himself the *tour de force* of rediscovering and proving the results of the whole generation of mathematicians who had worked on Hajos's conjecture (the connection was first noticed by M. Andreatta [9]). The first main result in Thm. 4 below is proved by providing a counter-example, the second is quite difficult:

Definition 1. A Vuza canon⁶ is a counterexample to Hajos's conjecture, i.e. a rhythmic canon $\mathbb{Z}_n = A \oplus B$ where neither A nor B is periodic.

I would like to point out that the notion of Vuza canons is musical, inasmuch as a canon with (say) a periodic outer voice is heard as the repetition of a shorter canon (with a shorter outer voice). This leads to a useful decomposition process, as we will see later.

Theorem 4.

- (1) *There exists Vuza canons.*
- (2) *Vuza canons only exist for periods n which are not of the form*

$$n = p^\alpha, n = p^\alpha q, n = p^2 q^2, n = p^2 qr, n = pqrs$$

where p, q, r, s are different primes.

A $\mathbb{Z}_n$ with n of the form above is often called, after Hajos, a “good group”, the other cyclic groups are “bad”. The smallest bad group is $\mathbb{Z}_{72}$, the next ones occur for $n = 108, 120, 144, 168, 180 \dots$ ⁷. Production of Vuza canons is discussed below, especially in section 2.3.

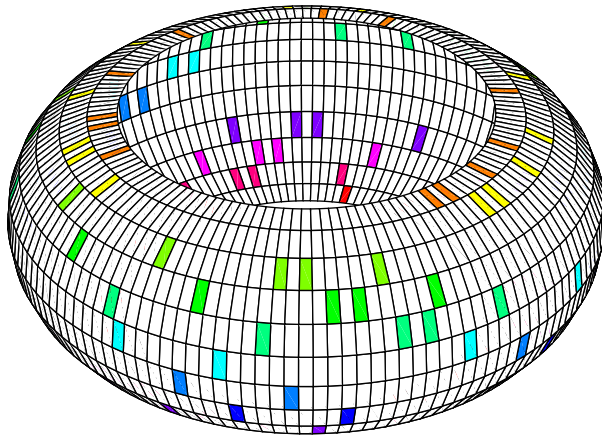


FIGURE 6. A Vuza canon with period 108 pictured on a torus.

⁶In some older papers, this term specifies those canons provided by Vuza's algorithm; this is no longer the case and we call 'Vuza canons' what he himself called 'Rhythmic Canons of Maximal Category'.

⁷Sloane's sequence A102562.

2.1.2. Transformations of rhythmic canons. Before we list the different transformational tools in use, it is necessary to pin down what is meant by identical canons. For instance, since the outer voice B is actually heard as $B + n\mathbb{Z}$ musically, there are many equivalent possible definitions of B . Also, notice that if $A \oplus B = \mathbb{Z}_n$ then any translate of A will tile with the same B (and similarly for translates of B). So we are really interested in classes of subsets of $\mathbb{Z}_n$, modulo translation. A common way to take this into account is to consider, instead of a subset, the sequence of consecutive differences between its elements; then the *basic form* of set A ⁸ is a fixed choice among the circular permutations of this interval sequence, usually inverse lexicographical order, see [1, 9] for instance and the definition below. Another formulation of this reduction to one representative of an orbit is the (computer science) concept of *Lyndon word*, which is instrumental in the enumeration of canons achieved by Friperntinger ([17]).

Definition 2. *The basic form of $A \in \mathbb{Z}_n$ is the smallest (for inverse lexicographic order) circular permutation of the set of consecutive intervals in A , $\Delta(A) = (a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, a_1 - a_k)$ where $0 \leq a_1 < a_2 < \dots < a_k < n$ are the elements of A , considered as numbers in $\llbracket 0, n-1 \rrbracket$.*

The basic form of a canon is the pair constituted by the basic forms of its inner and outer voices.

For instance, motif $A = \{0, 1, 4, 5\}$ in $\mathbb{Z}_8$ has basic form $(1, 3, 1, 3)$.

It is worthy of note that $\Delta(A + p)$ is a circular permutation of $\Delta(A)$. Notice also that the number of different motifs with the same basic form decreases when the motif has inner periodicities (motifs of Vuza canons are hence maximal in that respect, see section 2.3.1 below).

As discussed *supra*, the reduction of the *inner* voice A to a subset of $\mathbb{Z}_n$ is, perceptively speaking, a little dubious: $\{0, 1, 19, 34\}$ sounds quite differently from $\{0, 1, 2, 3\}$. But this is a price to pay for modelization. We will see below that the inverse transformation – from reduction to a larger motif – is the key to some enumeration problems.

Particularly after the publication of Vuza's work, several composers have been practising with rhythmic canons. These manipulations were formalized for implementation in musical softwares, like *OpenMusic*, and led to the following toolbox:

Duality is the exchange between inner and outer voice, i.e. from $A \oplus B = \mathbb{Z}_n$ we build $B \oplus A = \mathbb{Z}_n$ instead.

Affine transformation is probably the less obvious of all transformations of rhythmic canons (though it was rediscovered several times by non-mathematicians):

Theorem 5. *For any canon $A \oplus B = \mathbb{Z}_n$, for any affine transformation $f : x \mapsto ax + b \pmod n$ (meaning a is coprime with n), the affine transform of A by f still tiles with B , i.e.*

$$(aA + b) \oplus B = \mathbb{Z}_n$$

The proof is essentially Galois theory, as discerned already by Vuza who used this as a lemma (see also [4]). This transformation enables to change the motif (inner voice) without modifying the schedule of its entries (outer voice), or the reverse. On a more theoretical side, it allows a more compact classification of Vuza canons: for instance, there are only two different Vuza canons of period 72 up to affine transformation, $A = \{0, 3, 6, 12, 23, 27, 36, 42, 47, 48, 51, 71\}$ or $A' = \{0, 4, 5, 11, 24, 28, 35, 41, 47, 48, 52, 71\}$ with $B = \{0, 8, 10, 18, 26, 64\}$ instead of 6 inner voices and 3 outer voices, in basic form. We will see below that this feature is not unrelated to Z-relation and kindred topics.

Concatenation is the simplest transformation of all: it consists in replacing the motif by itself, repeated several times. In other words, $A \in \mathbb{Z}_n$ turns into $\overline{A}^k = A \oplus \{0, n, 2n, \dots, (k-1)n\} \in \mathbb{Z}_{kn}$. Strangely enough, the aural effect is very similar if the same transformation is applied to B instead. It is easy to check that

Theorem 6. *A tiles with B if and only $\overline{A}^k$ tiles with B .*

⁸Inspired of course of Forte's notion, see [30].

Let us notice that a Vuza canon is precisely a canon that *cannot* be produced by concatenation of some smaller canon. This enabled to reduce several conjectures⁹ or features of canons to Vuza canons, since:

Proposition 1. *Any canon can be produced by concatenation (and duality) from either the trivial canon $\{0\} \oplus \{0\}$, or a Vuza canon.*

Moreover, this entails a recursive construction of *all* tilings of finite ranges $\llbracket 0, n-1 \rrbracket$ (i.e. without reduction modulo n), since

Theorem 7. *Any canon $A \oplus B = \llbracket 0, n-1 \rrbracket$ can be reduced by concatenation and duality to the trivial canon.*

This was proved by G. deBruijn in [14].

Example 3. $\{0, 1, 4, 5\} \oplus \{0, 2\} = \llbracket 0, 7 \rrbracket$ is concatenated from $\{0, 1\} \oplus \{0, 2\} = \llbracket 0, 3 \rrbracket$, this last from $\{0, 1\} \oplus \{0\} = \llbracket 0, 1 \rrbracket$ which is a duplication of the trivial canon $\{0\} \oplus \{0\} = \llbracket 0, 0 \rrbracket$.

Other cases of reducible canons include the ‘assymmetric rhythms’ of [20], whose study originates in ethnomusicology.

Zooming and stuttering are two dual transformations. I called *stuttering* the act of replacing each note or rest in the motif by k repetitions of itself.

Example 4. For instance, from $\{0, 1, 4, 5\} \oplus \{0, 2\} = \mathbb{Z}_8$ one gets $\{0, 1, 2, 3, 4, 5, 12, 13, 14, 15, 16, 17\} \oplus \{0, 6\} = \mathbb{Z}_{24}$.

Algebraically, this means turning A into $\text{Stut}(A, k) = kA \oplus \{0, 1, 2 \dots k-1\} \in \mathbb{Z}_{kn}$. This time, in order to keep a canon it is necessary to *augment* the outer voice B into kB , i.e.

Theorem 8. *A tiles with B if and only $\text{Stut}(A, k)$ tiles with kB .*

Quite contrary to concatenation, these operations preserve the non-periodicity of either voice, and hence turn a Vuza canon into a (larger) Vuza canon. This has been used (in combination with the other transformations) in order to produce hitherto unknown Vuza canons, before Harald Friepertinger managed to enumerate all of them for periods 72 and 108 ([17]). Of course, it is equally possible to zoom on A and stutter with B .

Multiplexing is simple a extension of stuttering: instead of building $kA \oplus \{0, 1, 2 \dots k-1\}$, one chooses k inner voices $A_0, \dots A_{k-1}$ which tile with the same outer voice B , i.e. $A_0 \oplus B = A_1 \oplus B = \dots = \mathbb{Z}_n$, and the new motif with period kn is $\tilde{A} = \bigcup_{i=0}^{k-1} (kA_i + i)$. Again,

Theorem 9. $\tilde{A} \oplus kB = \mathbb{Z}_{kn} \iff \forall i = 0 \dots k-1, A_i \oplus B = \mathbb{Z}_n$.

This transformation (borrowed from a rather abstruse mathematical paper on tilings, [24]) opens interesting compositional possibilities, since several canons merge into a larger one while remaining audible. The dual transformation (multiplexing the outer voice) enlarges the motif and complexifies its outer voice.

An interesting theoretical aspect is that a kind of reciprocal stands: each canon wherein the outer voice can be written kB is multiplexed from a canon k times smaller (see on picture 7 how the smaller canons can be retrieved from the larger one). It was conjectured, in various contexts and by several authors, that essentially all canons were instances of some such multiplexing; but this is not true, as demonstrated by [38], though the smallest known counter-examples have period 900, see below subsection 2.3.5.

Uplifting This last transformation came to light in the latest developments of the search for Vuza canons [25]. It stems from a simple idea:

⁹Notably Fuglede’s conjecture, see [4, 5] for instance.

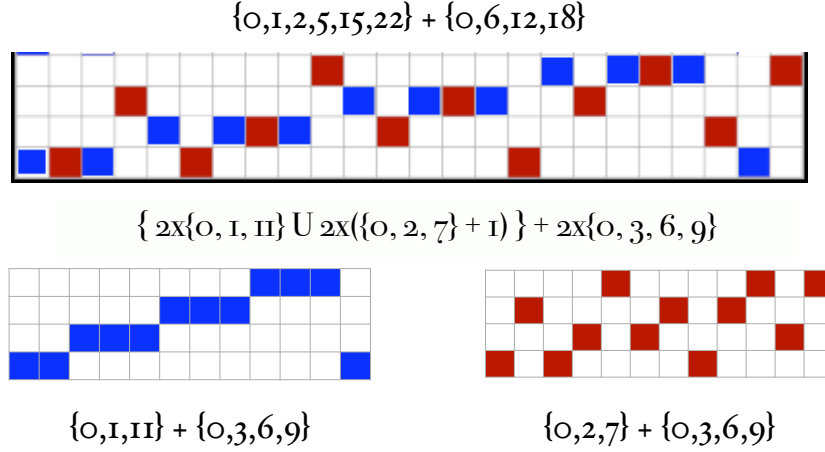


FIGURE 7. An example of multiplexing.

Proposition 2. *If A tiles $\mathbb{Z}_n$ then A tiles¹⁰ any larger cyclic overgroup $\mathbb{Z}_{kn}$; moreover, translating any element of A by any multiple of n provides a motif that also tiles $\mathbb{Z}_{kn}$.*

Proof. If $A \oplus B = \mathbb{Z}_n$, let $\tilde{A} = \{a_1 + k_1n, \dots, a_p + k_pn\} \subset \mathbb{Z}_{kn}$ where $A = \{a_1, \dots, a_p\} \subset \mathbb{Z}_n$ (this makes sense, since using the canonical projection Π from $\mathbb{Z}_{kn}$ to $\mathbb{Z}_n$ yields $\Pi(a + kn) = a$). Let also $\tilde{B} = \{b_i + \kappa n, b_i \in B, \kappa = 0 \dots k-1\}$; then it is straightforward to check that

$$\tilde{A} \oplus \tilde{B} = \mathbb{Z}_{kn}$$

considering that the sum mapping $\tilde{A} \oplus \tilde{B} \ni (a, b) \mapsto a + b$ is still injective and that $\#\tilde{A} \oplus \#\tilde{B} = kn$. $\square$

For instance, from $\{0, 1, 4, 5\} \oplus \{0, 2\} = \mathbb{Z}_8$ one uplifts the Bloch canon in example 3, e.g.

$$\{0, 9 = 1 + 8, 4, 5\} \oplus \{0, 2, 8, 10\} = \mathbb{Z}_{16}.$$

This is probably what Bloch actually did in order to produce his canon. But the main strength of this transformation is made clear when one is looking for some motif $A \in \mathbb{Z}_n$ knowing that A also tiles a smaller group. This was instrumental in many cases in the quest for all the smallest Vuza canons, see below in 2.2.4.

2.1.3. Length. It is of course of vital interest for a musician to predict the size of a rhythmic canon. The one obvious piece of information is the diameter of the motif, i.e. $\delta = \max A - \min A$. Unfortunately, this is of little use, since the overall length (period) n of the canon can vary widely. It is fairly easy to get $n = 1 + \delta$ (cf. $A = \{0, 2, 4 \dots n-2\}$ for some even n) and $n = 2\delta$, for $A = \{0, \delta\}$. Though Thm. 2 only yields $n \leq 2^\delta$, it was long thought that this last case $n = 2\delta$ was the upper limit, but this is not true.

Kolountzakis first proved [23] that n can be a non linear function of δ , with the construction given below in 2.3.6; it was later proved that n may be non polynomial in δ – larger than $e^{c\sqrt{\delta}}$ for some constant c . On the other hand, the upper bound was lowered to $n \leq e^{c\sqrt{\delta \ln \delta}}$, a notable improvement on $2^\delta = e^{\delta \ln 2}$ [31] though probably not optimal.

2.2. Algebraic modelizations and advanced tools.

¹⁰This sounds ambiguous, when A is considered as a part of $\mathbb{Z}_n$ not of $\mathbb{Z}$; but the point of the proposition is precisely that any subset of the integers reducing to A modulo n will also tile when reduced modulo some multiple kn of n .

2.2.1. *The zero set of the Fourier transform.* Another way to look at canons is by way of the characteristic functions (or ‘distributions’) of the voices, defined by $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else.} \end{cases}$.

This allows to take into account multisets, where notes can be repeated (and $1_A(x)$ is the number of superposed notes) or even rational- or real-valued maps, where the value may stand for volume for instance (the ‘Velocity’ component in MIDI format).

In terms of characteristic maps, the definition of a rhythmic canon by translation involves the *convolution product*:

Proposition 3.

$$A \oplus B = \mathbb{Z}_n \iff 1_A * 1_B = 1_{\mathbb{Z}_n} = \mathbf{1}$$

(the constant map equal to 1 for any element of $\mathbb{Z}_n$)

where $(f * g)(x) = \sum_{k \in \mathbb{Z}_n} f(k)g(x-k)$: clearly, in the case of characteristic functions, $1_A(k)1_B(x-k) = 1$ whenever $k \in A$ and $x-k \in B$, i.e. x is sum of an element of A and an element of B , and 0 else.

The main interest of this admittedly cumbersome product is that it turns into ordinary product after Fourier transform. This result is classic:

Proposition 4. If $\hat{f} : x \mapsto \sum_{k \in \mathbb{Z}_n} f(k)e^{-2i\pi kx/n}$ stands for the Fourier transform of map $f \in \mathbb{C}^{\mathbb{Z}_n}$, then

$$\widehat{f * g} = \hat{f} \times \hat{g}$$

Combining with the definition of tiling by equation (E), we get

Theorem 10.

$$A \oplus B = \mathbb{Z}_n \iff \widehat{1_A} \times \widehat{1_B} = n \widehat{1_{\mathbb{Z}_n}} = x \mapsto \begin{cases} n & \text{for } x = 0 \\ 0 & \text{else} \end{cases}$$

Essentially, setting apart the case of 0, the product of the Fourier transform of the characteristic maps of the inner and outer voices must be nil. This motivates the following definition:

Definition 3. $Z(A) = \{k \in \mathbb{Z}_n, \widehat{1_A}(k) = 0\}$ is the set of zeroes of the Fourier transform of [the characteristic map of] A .

With this definition,

Proposition 5. A tiles with outer voice B if and only if $Z(A) \cup Z(B) = \mathbb{Z}_n \setminus \{0\}$ and $\#A \times \#B = n$.

The zeroes of the Fourier transforms of A and B must cover $\mathbb{Z}_n$ (minus 0). For instance, with $n = 24$ and

$$A = \{0, 3, 12, 15\} \oplus B = \{0, 4, 8, 10, 14, 18, 26\}$$

we have

$$Z(A) = \{2, 6, 8, 24\} \quad \text{and} \quad Z(B) = \{3, 4, 6, 12\}$$

This last proposition is actually the fashionable definition for tilings among mathematicians [24]. We will see in the algorithms section how efficient it may prove in the quest for Vuza canons, for instance. A pretty corollary is the following:

Theorem 11. If A tiles with B , then so does any motif A' homometric with A .

The proof is straightforward when one recalls that ‘homometric’ (i.e. sharing the same interval content) is equivalent to sharing the same absolute value of the DFT, and hence the same $Z(A)$.¹¹ It remains to find significant examples of this result, with A' not congruent to A modulo transposition or inversion (i.e. Z -related in Forte’s definition), these transformations being special cases of Thm. 5.

¹¹The intervalic distribution of $A \in \mathbb{Z}_n$ is shown in $1_A * 1_{-A}$, whose Fourier transform is $\widehat{1_A} \times \widehat{1_{-A}} = |\widehat{1_A}|^2$.

Using $Z(A)$ is not as cumbersome as it looks: the definition by covering of $\mathbb{Z}_n \setminus \{0\}$ is less fuzzy than it seems, since these sets of zeroes are not arbitrary subsets of $\mathbb{Z}_n$; they admit a large degree of organization:

Theorem 12. *For any set $A \subset \mathbb{Z}_n$, if $x \in Z(A)$ then $Z(A)$ contains all elements of the group $(\mathbb{Z}_n, +)$ sharing the order of x , i.e. all multiples of x by any integer coprime with n .*

Recall the order of $x \in \mathbb{Z}_n$ is the smallest integer m such that $mx = 0 \pmod n$, hence $m = n/\gcd(n, x)$. For instance, with B in the example above, 15 has order $72/3 = 24$ and $Z(B)$ contains 3, 15, 21, 33, 39, 51, 54, 69 which are all of order 24. A shorter description of $Z(B)$ is “all elements of $\mathbb{Z}_{72}$ with order 3, 4, 6, 12, 24, or 36”. This explains why mA tiles with B whenever A does: the theorem really states that $Z(mA) = Z(A)$. We will see below how this shorter description of $Z(A)$ is instrumental in some practical algorithms for rhythmic canons: the set of these orders is precisely the set R_A defined below.

2.2.2. Polynomials. This last theorem is not obvious, it is better understood with the equivalent formulation of tiling in terms of polynomials. Recall the expression of

$$\widehat{1}_A(x) = \sum_{k \in A} e^{-2i\pi kx/n} = \sum_{k \in A} e^{-2i\pi x/n^k}$$

which appears as a polynomial in $\xi^x = e^{-2i\pi x/n}$, the generic n^{th} root of unity.

Definition 4. *The characteristic polynomial of subset $A \subset \mathbb{Z}_n$ is $A(X) = \sum_{k \in A} X^k$.¹²*

Then $\widehat{1}_A(x) = A(e^{-2i\pi x/n})$. Conversely, knowing all n values of $\widehat{1}_A$ completely determines the polynomial $A(X)$, since its degree is $< n$. Now the equation for tiling becomes

Theorem 13.

$$A \oplus B = \mathbb{Z}_n \iff A(X) \times B(X) = 1 + X + X^2 + \dots X^{n-1} \pmod{(X^n - 1)} \quad (T_0)$$

This can be checked directly from the definition of $A(X)$, or derived from the product of Fourier transforms. This is the traditional tool for the studies of tilings, from the seminal work of Redei, Hajos, deBruijn *et alii* in the fifties, to the late nineties. It is worthy of note that tiling reduces to factoring a very special polynomial:

$$\mathbf{1}(X) = 1 + X + \dots X^{n-1} = \frac{X^n - 1}{X - 1}$$

but in the non factorial ring $\mathbb{Z}[X]/(X^n - 1)$. This explains both the immense variety of rhythmic canons,¹³ and the difficulty of the problem¹⁴.

The factors of $\mathbf{1}(X)$ in $\mathbb{Z}[X]$ are well known, they are the cyclotomic polynomials (so called because they partition the roots of unity on the unit circle):

Definition 5. *The k^{th} cyclotomic polynomial is $\Phi_k(X) = \prod (X - \xi)$ where ξ runs over the set of primitive k^{th} roots of unity (i.e. $\xi^k = 1$ but $\xi^m \neq 1$ for $0 < m < k$). They can all be computed from the functional identity*

$$\prod_{d|n} \Phi_d(X) = X^n - 1$$

¹²This can, and will, be extended to the case of a multiset.

¹³Kolountzakis proved that the number of different possible outer voices B is larger than $e^{C\sqrt{n}}$, for arbitrarily large values of n , and this even in the case of Vuza canons [25]. For instance, the extremely regular motif $A = \{0, 10, 20, 30, 40\}$ admits already 195,375 different complements modulo 50 (up to translation), which begins to get inconvenient for practical computation.

¹⁴The comparison with prime number factorization and Diophantine equations in [34] thus rather underplays the difficulty. But fortunately the very imperfection of the ambient ring of polynomials opens up new alleys of exploration, with the adequate mathematical tools.

The simplest ones arise for $k = p$, prime: $\Phi_p(X) = 1 + X + X^2 + \dots + X^{p-1}$. Both Thms 12 and 5 derive from the irreducibility of these cyclotomic factors. Since equation 13 can be rewritten in $\mathbb{Z}[X]$ as

$$A(X) \times B(X) = \mathbf{1}(X) + Q(X)(X^n - 1) = \mathbf{1}(X)(1 + (X - 1)Q(X)) = (1 + (X - 1)Q(X)) \prod_{1 < d|n} \Phi_d(X)$$

the irreducibility of the Φ_d 's entails that each Φ_d must divide either $A(X)$ or $B(X)$ (even though $A(X)$ is really only defined modulo $X^n - 1$ – see above the reduction from $A \in \mathbb{Z}$ to $A \in \mathbb{Z}_n$). Hence the central role played by these cyclotomic factors, which culminated with the conditions for tiling found by Coven and Meyerowitz in the last years of the century.

2.2.3. *Conditions (T_1) and (T_2) .* In [11] they introduced, for $A \subset \mathbb{Z}_n$,

Definition 6. $R_A = \{d, \Phi_d \mid A(X)\}$ and $S_A = \{p^\alpha \in R_A, p \text{ prime}\}$.

For instance with $A = \{0, 3, 6, 12, 23, 27, 36, 42, 47, 48, 51, 71\}$ one gets $R_A = \{2, 8, 9, 18, 72\}$, $S_A = \{2, 8, 9\}$.¹⁵ The presence of all the $\Phi_d, d \mid n$, in $A(X) \times B(X)$ entails that

- $S_A \cup S_B$ is the set¹⁶ of all prime powers dividing n , and
- $R_A \cup R_B$ is the set of all divisors of n (1 excepted).

Coven and Meyerowitz then proceeded to prove the following statements, the last of which is quite difficult.

Theorem 14. *Defining conditions*

$$(T_1): \prod_{p^\alpha \in S_A} p = \#A;$$

$$(T_2): \forall p^\alpha, q^\beta, r^\gamma \dots \in S_A, p^\alpha q^\beta r^\gamma \dots \in R_A \text{ (products of powers of distinct primes belonging to } S_A \text{ are in } R_A);$$

one has

- (1) If A tiles, then (T_1) is true.
- (2) If both $(T_1), (T_2)$ are true, then A tiles.
- (3) If $\#A$ has at most two different prime factors, and A tiles, then both $(T_1), (T_2)$ are true.

As of today, it is not known whether condition (T_2) is always necessary for tiling. With the example above we can check $(T_1) : \#A = 12 = 2 \times 2 \times 3$ since $S_A = \{2^1, 2^3, 3^2\}$, and $(T_2) : 2 \times 9 \in R_A$ and $8 \times 9 \in R_A$.

It is important in practice to note that if A tiles, then it tiles already with period $\text{lcm}(S_A)$ (reducing A to a subset of $\mathbb{Z}_{\text{lcm}(S_A)}$).

The link with Fourier transforms is straightforward: recall the organization of $Z(A)$ in subsets of elements with equal orders, these orders are precisely the elements of R_A . For instance when $R_A = \{2, 8, 9, 18, 72\}$ we have $Z_A = \{36\} \cup \{9, 45, 63\} \cup \{8, 16, 32, 40, 56, 64\} \cup \dots$, the multiples¹⁷ of n/d where d runs over R_A .

Another essential feature of rhythmic canons is apparent on R_A : the *periodicity* of A can be checked on either $Z(A)$ or R_A , and this is an important part of the most recent algorithms used for the search of Vuza canons.

¹⁵Actually their definition stands for $A \subset \mathbb{Z}$; we simplify slightly their exposition, since for any other polynomial congruent with $A(X) \pmod{(X^n - 1)}$, the subset of the divisors of n in R_A , which are the indexes of the relevant cyclotomic factors, does not change. We choose this as our definition for R_A . Anyhow, S_A is always made of divisors of n .

¹⁶They show that corresponding cyclotomic polynomials occur only once, so this is a partition of the set of all prime powers dividing n . On the other hand, sometimes $R_A \cap R_B \neq \emptyset$.

¹⁷Multiple by some invertible, i.e. a number coprime with n , cf. supra. Beware that in the example given, 16 is not 8×2 but $8 \times 11 \dots$

Theorem 15. *A is periodic in $\mathbb{Z}_n$ if and only if the complement set of $Z(A)$ is part of some subgroup of $\mathbb{Z}_n$. In practice, one may check whether there exists a prime factor of n which divides all elements not in $Z(A)$.*

This can be checked almost visually. For $A' = \{0, 5, 8, 13\}$, which tiles $\mathbb{Z}_{16}$, $R_{A'} = S_{A'} = \{2, (10), 16\}$ and (keeping $n = 16$) $Z(A') = \{1, 3, 5, 7, 8, 9, 11, 13, 15\}$ contains the complement of subgroup $2\mathbb{Z}_{16}$, hence is $16/2 = 8$ -periodic. The non zeroes are clearly members of a subgroup on picture 8.

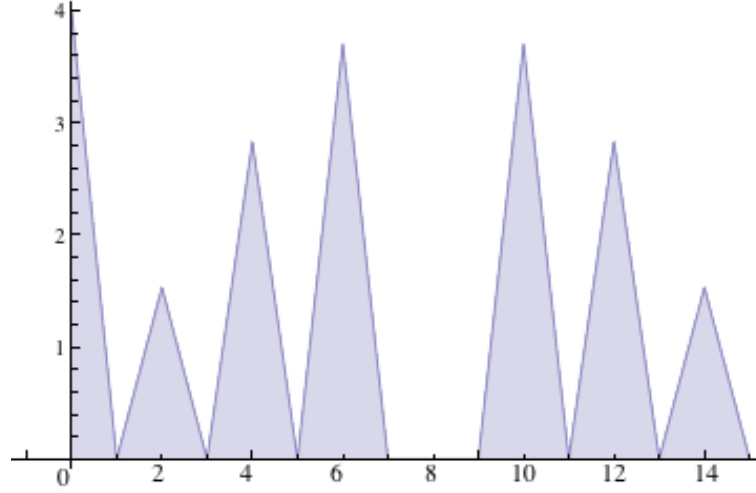


FIGURE 8. The complement of $Z(A')$ is part of $2\mathbb{Z}_{16}$ as seen on the graph of $|\widehat{1_{A'}}|$.

This theorem can be proved quite easily owing to the fact that a subset with period smaller than n can be described with fewer than n Fourier coefficients. In the example above, $A' = A'' \oplus \{0, 8\}$ where $A'' = \{0, 5\}$, and we recognize the kinship between their respective Fourier transforms on picture 9. A

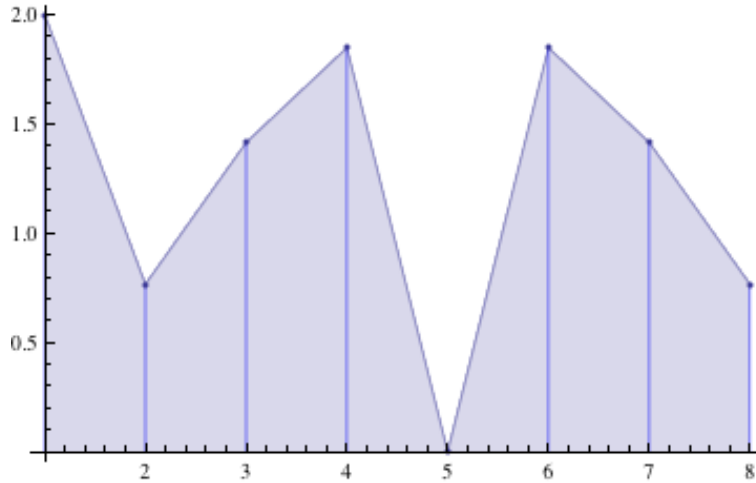


FIGURE 9. The Fourier transform of A'' .

useful special case is the following

Lemma 1. *If A is a metronome: $A = \{0, r, 2r, \dots, (k-1)r\}$ whose doubled length divides n (i.e. $2k \mid n$) then any complement B of A is periodic.*

This enabled to discard motifs like $\{0, 10, 20, 30, 40, 50\}$ in the search for Vuza canons in $\mathbb{Z}_{180}$.

2.2.4. Matolcsi's algorithm. After meeting in Ircam with several actors on the rhythmic canons scene, Mate Matolcsi proposed an algorithm for the exhaustive search of Vuza canons in $\mathbb{Z}_n$, which is detailed in the algorithms section below. The general philosophy is to use a toponomy of motifs based on the set

of cyclotomic indexes R_A ¹⁸, which generalizes the classification up to translation, retrogradation and even affine transform (all of which do not change R_A). In practice, it had to be completed with several tricks of the trade (see [5, 25]). This algorithm allows, more generally, to find all canons with given S_A, S_B , should one remove the conditions on non-periodicity. It enabled to complete the classification of Vuza canons initiated by Friperinger [16] for $n = 72, 108$ to $n = 120, 144, 168$. As a by-product, I was able to prove that condition (T_2) is indeed necessary for tiling, for all canons of period $n \leq 180$, in finding all Vuza canons for $n = 120, 168$.¹⁹

The results of Friperinger, Amiot, Kolountzakis and Matolcsi for the enumeration of Vuza canons up to period 168 are summed up in the following table (fig. 10). Factors are enumerated up to circular permutation.

n	R_A	R_B	number of A's	number of B's
72	{2, 8, 9, 18, 72}	{3, 4, 6, 12, 24, 36}	6	3
108	{3, 4, 12, 27, 108}	{2, 6, 9, 18, 36, 54}	252	3
120	{2, 3, 6, 8, 15, 24, 30, 120}	{4, 5, 10, 12, 20, 40, 60}	20	16
120	{2, 5, 8, 10, 15, 30, 40, 120}	{3, 4, 6, 12, 20, 24, 60}	18	8
144	{2,8,9,16,18,24,72,144} or {2,8,9,16,18,72,144}	{3,4,6,12,24,36,48}	36	6
144	{2, 4, 9, 16, 18, 36, 144} or {2, 4, 6, 9, 16, 18, 36, 144} or {2, 4, 9, 12, 16, 18, 36, 144}	{3,6,8,12,24,36,72}	8640	3
144	{3, 4, 6, 8, 12, 24, 48, 72} or {3, 4, 6, 8, 12, 24, 36, 48, 72}	{2,9,16,18,144} or {2,9,16,18,36,144}	156 +6	48 +12
144	{2, 3, 6, 8, 12, 24, 48, 72} or {2, 3, 6, 8, 12, 18, 24, 48, 72}	{4, 9, 16, 18, 36, 144}	324	6
168	{3, 4, 6, 12, 24, 28, 84}	{2, 7, 8, 14, 21, 42, 56, 168}	16	54
168	{4, 7, 12, 14, 28, 26, 84}	{2, 3, 6, 8, 21, 24, 42, 168}	104	42

FIGURE 10. Classification of Vuza canons according to their set R_A .

2.2.5. Tiling the line and differences. There is a strange and deep relationship between the question of tilings and the set of differences $A - A = \{a - a', (a, a') \in A \times A\}$. We have already met a relationship between homometry and tiling (Thm. 11), where the interval vector is nothing but $A - A$ considered as a multiset. It is also possible to define a tiling by differences: in [17] for instance the following definition is given:

Definition 7. *A tiles $\mathbb{Z}_n$ if there is a set of translates of $A, A_i = A + t_i, i = 1 \dots p$ such that for $U = \cup_{i=1}^p A_i$, one has $U - U = \mathbb{Z}_n$.*

This is more akin to the definition of a *covering* that we gave in the first section. For instance, Friperinger gives as an example $n = 8, A = \{0, 4, 5, 7\}$ with outer voice $B = \{0, 4, 6\}$, whose sum is the multiset $\{0, 0, 1, 2, 3, 3, 4, 4, 5, 5, 6, 7\}$. We get a mosaic canon when U is actually equal to $\mathbb{Z}_n$.

Perhaps this stemmed from the following simple characterization:

Proposition 6. $A \oplus B = \mathbb{Z}_n \iff \#A \times \#B = n$ and $(A - A) \cap (B - B) = \{0\}$.

Proof. The second condition means that the sum is direct, since

$$a + b = a' + b' \iff a - a' = b' - b$$

and the first one ensures that this direct sum fills the whole of $\mathbb{Z}_n$. □

¹⁸Because $R_A = R_{A'}$ and $A \oplus B = \mathbb{Z}_n$ implies $A' \oplus B = \mathbb{Z}_n$.

¹⁹The case $n = 144$ treated by [25] could not yield counterexamples, since 144 has only two prime factors and Coven-Meyerowitz had proved that in such tilings (T_2) must be satisfied. The same goes for any motif tiling $\mathbb{Z}_{180}$.

Another, deeper connection with difference sets is the *spectral conjecture*. In its general form, it is stated as “tiling $\iff$ spectral”, where ‘spectral’ means that the tile (a measurable subset of $\mathbb{R}^n$ in the most general context) admits a Hilbert basis of exponential functions, just like any map in $L^2([0, 1])$ is the sum of its Fourier expansion. In our context, we have a less esoteric definition involving difference sets:

Definition 8. *A subset $A \in \mathbb{Z}$ is spectral if there exists a spectrum $\Lambda \subset [0, 1[$, i.e. a subset with the same cardinality as A , such that $e^{2i\pi(\lambda_i - \lambda_j)}$ is a root of $A(X)$ for all distinct $\lambda_i, \lambda_j \in \Lambda$.*

The spectral conjecture has been proved in many cases (convex tiles for instance) but in general it is false, as first shown in high dimension by Field medalist Terence Tao [39]. Following further work [24], the conjecture only remains open in dimensions 1 et 2. In dimension 1 which is our context for rhythmic canons, Izabella Łaba has proved that $(T_1) + (T_2)$ implies ‘spectral’, just as [11] proved it implied ‘tiling’. So the conjecture is known to be true when n has only two prime factors, by the last result in Thm. 14, and also for motifs that tile a ‘good group’ (non Hajos) [3]. The reduction techniques by transformations of canons also prove that this conjecture is true if and only if it is true for Vuza canons (see [1, 5, 18]), which is another stringent motivation for their study.

2.3. Some algorithms.

2.3.1. *Vuza canons.* Numerous algorithms have been devised by various mathematicians for computing what we call Vuza canons in the scope of this article. Some are fairly cumbersome. For an exposition and comparison of the different methods, see [15]. For a reasonably efficient program aiming at a complete classification for a given period, see paragraph 2.3.3 below. For those who are averse to complicated programming, the best solution is probably to make use of already computed tables.²⁰ Using some of the transformational techniques described in this article enriches the field of solutions considerably.

As a starting point for constructing large, uncatalogued Vuza canons, we give the following algorithm, which is the simplest (providing just one solution); it was exposed by Franck Jedrzejewski [22]:

Proposition 7 (2003). *Let p_1, p_2 be prime numbers and $n_i, i = 1..3$ satisfying $n_1 p_1$ coprime with n_2 and $n_2 p_2$ coprime with n_1 ; then, denoting $\llbracket a, b \rrbracket = \{a, a + 1, \dots, b\}$ we construct for $n = p_1 p_2 n_1 n_2 n_3$ a Vuza canon $R \oplus S = \mathbb{Z}_n$, with:*

$$\begin{aligned} A &= n_2 n_3 \times (\llbracket 0, p_2 - 1 \rrbracket \oplus p_2 n_1 \times \llbracket 0, p_1 - 1 \rrbracket) & B &= n_1 n_3 \times (\llbracket 0, p_1 - 1 \rrbracket \oplus p_1 n_2 \times \llbracket 0, p_2 - 1 \rrbracket) \\ S &= n_3 (p_2 n_2 \times \llbracket 0, n_1 - 1 \rrbracket \oplus p_1 n_1 \times \llbracket 0, n_2 - 1 \rrbracket) & R &= (\llbracket 1, n_3 - 1 \rrbracket \oplus B) \cup A \end{aligned}$$

2.3.2. *Completion.* An essential part of the building of rhythmic canons is the search for the outer voice B , knowing the inner one A . This enables to enrich the catalog, starting from the Vuza canons provided by partial algorithms for instance. There are several ways to tackle this difficult problem (probably NP-complete, according to [25]). We begin with the more naive one.

Assume the motif A begins with 0. We build recursively a complement B , also beginning with 0.

B is initialized to $\{0\}$. We call $\text{complete}(A, \{0\})$ where the procedure $\text{complete}(A, B)$ is as follows:

- If $\#A \#B = n$, then add B to the list of complements of A ; exit.
- If not, consider the complement C of $A \oplus B$ in $\mathbb{Z}_n$ (by construction, this sum is direct).
For all $x \in C$, test whether $x + A \subset C$. If so, then call $\text{complete}(A, B \cup \{x\})$.

After this step, all solutions must be reduced to basic form so that doublons can be eliminated.

Many bells and whistles may be added. A sound trick (helping to reduce redundancy in the list of solutions) was used by Fripertinger: imposing the condition that the greatest interval in B is the first one, i.e. if its elements are sorted as $0 = b_0 < b_1 < b_2 < \dots < n$, then $\forall i \geq 0, b_{i+1} - b_i \leq b_1 - b_0$. This

²⁰Like <http://canonsrythmiques.free.fr/EnumerationVuzaCanons.rtf>

algorithm is still valuable for fairly regular, run-of-the-mill, motifs. When looking for Vuza canons for instance, we can use the more involved routine proposed by Matolcsi ([25]):

Again, B is initialized to $\{0\}$. We call $\text{oracle}(A, \{0\})$ where the procedure $\text{oracle}(A, B)$ is defined recursively as follows:

- If $\#A\#B = n$, then add B to the list of complements of A ; exit.
- If not, consider the complement C of $A \oplus B$ (again, the sum is direct).

For all $x \in C$, compute $r(x)$, the number of k such that simultaneously $\begin{cases} x - k \in A \\ (A + k) \subset C \end{cases}$.

- Sort the values of $r(x)$ and take $x = xMin$ achieving the minimal value for $r(x)$.
- If this minimal value $r(xMin)$ is 0, then exit (this branch of the tree is a dead end); else call $\text{oracle}(A, B \cup \{k\})$ for all the $r(xMin)$ values of k corresponding to $xMin$.

End as before by sorting all basic forms in the list of complements found.²¹

The point of complicating the algorithm is that, especially for irregular motifs, there may be very few ways to ‘fill in the hole’ at x , and this means fewer branches to explore in the tree (sometimes only one...or even none). This algorithm is particularly suited to the research of Vuza canons, especially in combination with the specific formula of Coven and Meyerowitz which proves the second point of Thm. 14:

Proposition 8. *If A satisfies (T_1) and (T_2) , then a complement of A in $\mathbb{Z}_n$, i.e. B satisfying $A \oplus B = \mathbb{Z}_n$, can be produced by its characteristic polynomial: $B(X)$ is the product of the $\Phi_{p^\alpha}(X^{n/p^{v(p)}})$, where $p^\alpha \mid n$ is not in S_A , and $n = \prod_i p_i^{v(p_i)}$ is the decomposition of n into prime powers (so that $n/p^{v(p)}$ is the largest divisor of n coprime with p).*

Example 5. Consider $S_A = \{2, 8\}$ and $n = 24$.²² Since $24 = 2^3 3^1$, the missing prime powers in S_A – which must indeed be in S_B –, are $4 = 2^2$ and 3 , which are respectively complemented to 24 by coprime prime powers 3 and 8; we compute

$$B(X) = \Phi_4(X^3) \times \Phi_3(X^8) = (1 + (X^3)^2)(1 + (X^8) + (X^8)^2),$$

hence $B = \{0, 6, 8, 14, 16, 22\}$, which does indeed tile with $A = \{0, 3, 12, 15\}$ for instance.

A third algorithm is the one used by Davalan, making use of graphs and maximal cliques, see [13] in this issue of *PNM*. It compares quite favorably with the other ones, especially for regular motifs, and it was indeed used for double-checking some of the results on Vuza canons with period 180.

2.3.3. Algorithms involving some form of completion. This construction gave birth to two applications. The first one is compositional: the software *OpenMusic* developed a patch (a kind of graphic user interface module) for creating so-called *cyclotomic canons* [12]. The recipe is straightforward:

- Choose n and $\#A$ (a divisor of n). The program will now deliver several (not all) rhythmic canons with this configuration without other input from the user.
- Compute all possible S_A , using (T_1) .
- For each possible S_A compute R_A using (T_2) .
- Compute $\prod_{k \in R_A} \Phi_k = A(X)$. Keep it whenever the coefficients are only 0’s and 1’s, else discard.
- Compute $B(X)$ from S_A with the Coven-Meyerowitz formula above.

²¹With an Intel Core duo processor at 2.66 GHz, complements in Vuza canons with periods between 100 and 200 are found in seconds; but the 195,375 complements of $(0, 10, 20, 30, 40)$ modulo 50 are found in a little more than one hour. Such regular ‘metronomes’ are particularly time-consuming. See [27] for an embryonic theory of tilings with similar motifs.

²²If we start from an actual motif A and n is unknown, Coven and Meyerowitz proved that n can be taken equal to the lcm of R_A – or any multiple thereof.

Example: $n = 24, \#A = 6$. Then S_A must contain exactly 3 and one power of 2, i.e. is one of the following: $\{2, 3\}, \{4, 3\}, \{8, 3\}$. The corresponding R_A 's are $\{2, 3, 6\}, \{4, 3, 12\}, \{8, 3, 24\}$. The product of cyclotomic polynomials yields respectively

$$X^5 + X^4 + X^3 + X^2 + X + 1, X^8 + X^7 + X^6 + X^2 + X + 1, X^{14} + X^{13} + X^{12} + X^2 + X + 1$$

As it happens, all are 0-1 polynomials. Corresponding B 's are computed with the Coven-Meyerowitz formula, respectively

$$X^{18} + X^{12} + X^6 + 1, X^{15} + X^{12} + X^3 + 1, X^9 + X^6 + X^3 + 1$$

Finally we get the three different rhythmic canons shown on fig. 11.

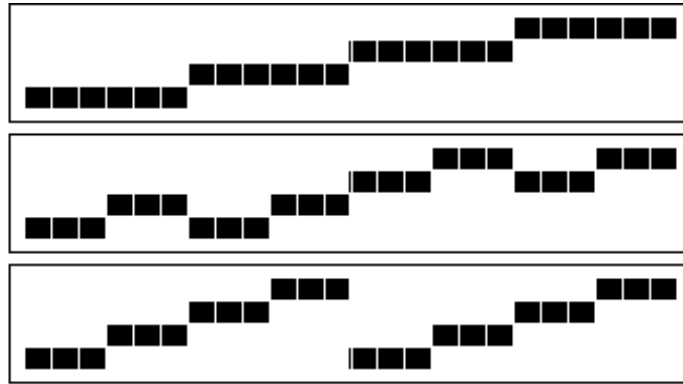


FIGURE 11. Three solutions for $n = 24, \#A = 6$.

The second application of the Coven-Meyerowitz completion formula is (part of) an algorithm for cataloguing Vuza canons in a given $\mathbb{Z}_n$ (a bad group), due to Mate Matolcsi [25, 5]). It enabled to check Friertinger's results for $n = 72, 108$ and to complete the catalog for $n = 120, 144$, and recently 168. The idea is to check for all possible sets S_A .

- Compute all partitions in two subsets of the set of prime power divisors of n . Keep (usually) the smallest part, which will be S_A (the other being of course S_B).
- Compute the Coven-Meyerowitz complement B for S_A .²³
- Compute all possible A completing B , using one of the general completion algorithms described above.²⁴ Sort by the different values of R_A , keeping one representative for each possibility.
- Discard all sets R_A that either
 - (1) ensure that A is periodic, or
 - (2) ensure that B must be periodic (recalling that R_B must contain at least all divisors of n not in R_A), making use of Thm. 15.
- For each remaining representative of possible A 's, compute complements B with **oracle**, discarding the periodic ones.
- Whatever remains is a Vuza canon.

Details and tables of results are given in [8].

Another construction advanced by the same authors remedied the excessive computer-time required for some cases. It makes full advantage of a remark above (to be found originally in [11]), that if A tiles $\mathbb{Z}_n$ then A also tiles $\mathbb{Z}_{\text{lcm}(S_A)}$. Often this is actually the best way to compute all complements of A , because the **oracle** procedure is not at its best on very regular motifs, such as those built up by the Coven-Meyerowitz completion formula. I will describe it on an example: the most difficult case for $n = 168$, computationally speaking, was $S_A = \{2, 3, 8\}$. Eventually I solved it by considering all possible values of A in $\mathbb{Z}_{24}$, i.e. all tilings of $\mathbb{Z}_{24} = \hat{A} \oplus B$ where $S_{\hat{A}} = \{2, 3, 8\}$. There are exactly 6 (basic forms

²³This is a periodic motif.

²⁴For a periodic, regular motif such as this B , **complete** is sometimes no slower than **oracle**.

of) solutions, found by completion of the Coven-Meyerowitz complement as explained above but with $n = \text{lcm}(2, 3, 8) = 24$. This complement is $(6, 18)$ (in basic form, meaning e.g. $\{0, 6\} \subset \mathbb{Z}_{24}$ or any of its translates) and its six complements, i.e. possible values of $\hat{A}$, are (up to translation)

$$\begin{aligned} &\{0, 1, 2, 3, 4, 11, 12, 13, 14, 15, 16, 23\}, \{0, 2, 3, 4, 7, 11, 12, 14, 15, 16, 19, 23\}, \\ &\{0, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23\}, \{0, 1, 2, 3, 5, 10, 12, 13, 14, 15, 17, 22\}, \\ &\{0, 2, 3, 5, 7, 10, 12, 14, 15, 17, 19, 22\}, \{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21\} \end{aligned}$$

Now it only remained for each of these values to test the (aperiodic) complements of all $\hat{A}$'s *uplifted* modulo 168, e.g.

$$A = \{0, 1, 2, 3, 4, 11, 12, 13, 14, 15, 16, 23\} + 24\{a, b, c \dots \ell\}, 0 \leq a, b \dots \ell \leq 6$$

(moreover, $a = 0$ may be assumed without loss of generality). Examination of these 72 million cases or so (taking the aperiodic A 's and looking for aperiodic complements) yielded 104×42 solutions in reasonable time.

Other theoretical results prove to be of practical importance. For instance, when trying to find Vuza canons as outlined above for $n = 168, S_A = \{4, 8\}$, the computer stumbled on the many million complements of $\{0, 28, 42, 70\}$. But since this motif is a multiple of 14, any canon $A \oplus B = \mathbb{Z}_{168}$ can be demultiplexed, i.e. $B = \cup(i + 14B_i)$ where B_i tiles $\mathbb{Z}_{168/14}$ with $A/14$, i.e. $B_i \oplus \{0, 2, 3, 5\} = \mathbb{Z}_{12}$. This yields $B_i = \{0, 4, 8\}$, (or any translate thereof) and hence B must be $4 \times 14 = 56$ -periodic.

2.3.4. Computation of $Z(A)$ and R_A . We mention this small subroutine, because it can be quite time-consuming if the programming is sloppy (from personal experience). M. Matolcsi suggested the following alternative to the polynomial division by some cyclotomic factor. The salient mathematical fact is that we need only check $A(X)$ for $X = e^{2i\pi/d}$ where d carries over the list of divisors of n (substantially shorter than the range $1 \dots n$, eg 16 cases only for $n = 168$). The algorithm runs as follows:

- Compute the list of divisors of n .
- For any d in this list, compute $A(e^{2i\pi/d}) = \sum_{k \in A} e^{2ik\pi/d}$. If necessary, increase precision (for n in the range of several hundreds, I use 20 digits floating arithmetic).
- If the result is zero (within a given tolerance), then all elements of $\mathbb{Z}_n$ with order d are in $Z(A)$, or equivalently, $d \in R_A$. The list of these specific values of the DFT can be recorded, normalized (0 or not) for comparison purposes, since it characterizes R_A or $Z(A)$.

We will not provide algorithms for integer arithmetic routines, like listing divisors (or selecting which are powers of prime numbers), which are sometimes pre-build in programming languages, and anyway are easy to write from scratch, say in C.

It *is* possible to build an algorithm with exact polynomial arithmetic - checking whether $A(X)$ is divisible by a given (tabulated) cyclotomic polynomial, but it proved to be way too long for applications like the computation of all Vuza canons of order 144. Davalan used a neat trick for the purpose of classification, consisting in computing the gcd of $A(X)$ and $X^n - 1$, thus retaining the product of all relevant cyclotomic factors.

2.3.5. Szabo's construction. This was advanced in [38] for building counterexamples to a conjecture of Sands (destroying in one fell swoop a number of other conjectures too). It provides rather large canons - the smallest known period working for this algorithm is $n = 900$, see [4].

- Choose some prime numbers $u_i, v_i, i = 1 \dots 3$;
- The period n is $n = \prod_{i=1}^3 u_i v_i$.
- With $g_i = \frac{n}{u_i v_i}, i = 1 \dots 3$, compute $A = \bigoplus_{i=1}^3 \{0, g_i, 2g_i, \dots, (u_i - 1)g_i\}$, a direct sum of arithmetic sequences with ratio g_i and length u_i . Hence $\#A = u_1 u_2 u_3$.

- Similarly we build a first version of $B = \bigoplus_{i=1}^3 \{0, u_i g_i, 2u_i g_i, \dots, (v_i - 1)u_i g_i\}$: so $\#B = v_1 v_2 v_3$. It is straightforward to check that $A \oplus B = \mathbb{Z}_n$, B being quite simple (a subgroup).²⁵ So we introduce a perturbation, producing fake banknotes:
- We choose some permutation $\sigma \in S_3$ (say $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$).
- Add to all elements of B with form $ku_i g_i + u_{\sigma(i)} g_{\sigma(i)}$ the value g_i (recall that general elements of B have the form $ku_1 g_1 + \ell u_2 g_2 + mu_3 g_3$, so we adress a small number of them here).
- The perturbed outer voice B' still tiles with A .

This yields Vuza canons, with the additional irregularity that they are not the result of the multiplexing of any smaller canon (this is guaranteed with some technical conditions on the u_i, v_i which are sufficient but fortunately not necessary).

The simplest case is when the u_i (resp. v_i) are 2, 3 and 5, yielding period 900 canons.

2.3.6. Kolountzakis's construction. The preceding algorithm smacks of a geometrical construction: A is build up like a direct sum of arithmetic sequences in three different 'directions'. This is explicit in the following algorithm, that Mikhalis Kolountzakis devised in order to prove that the period of the smallest canon tiled by A may be quite large, compared with the width of the motif A itself.

The starting point is that for $n = 30ab$ (where a, b are odd primes > 5), the cyclic group $\mathbb{Z}_n$ is isomorphic with the product $\mathbb{Z}_{3a} \times \mathbb{Z}_{5b} \times \mathbb{Z}_2$, which can in turn be seen as two identical sheets (the last value, in $\mathbb{Z}_2$, indexes the sheets), which are rectangles²⁶ tiled by ab identical 3×5 tiles. This elementary tile is A : in 3D coordinates, it is $\{0, 1, 2\} \times \{0, 1, 2, 3, 4\} \times \{0\}$. So B is initially the lattice $B = \{0, 3, 6, \dots, 3(a-1)\} \times \{0, 5, \dots, 5(b-1)\} \times \{0, 1\}$. Just as in Szabo's construction, B is perturbed so as to render it non periodic. Graphically, Kolountzakis suggested to shift a row in one sheet and a column in the other. Algorithmically, all combinations of such shifts can be tried (some will yield periodic canons, and may be discarded). It only remains to apply explicetely the isomorphism from $\mathbb{Z}_{3a} \times \mathbb{Z}_{5b} \times \mathbb{Z}_2$ to $\mathbb{Z}_n$.

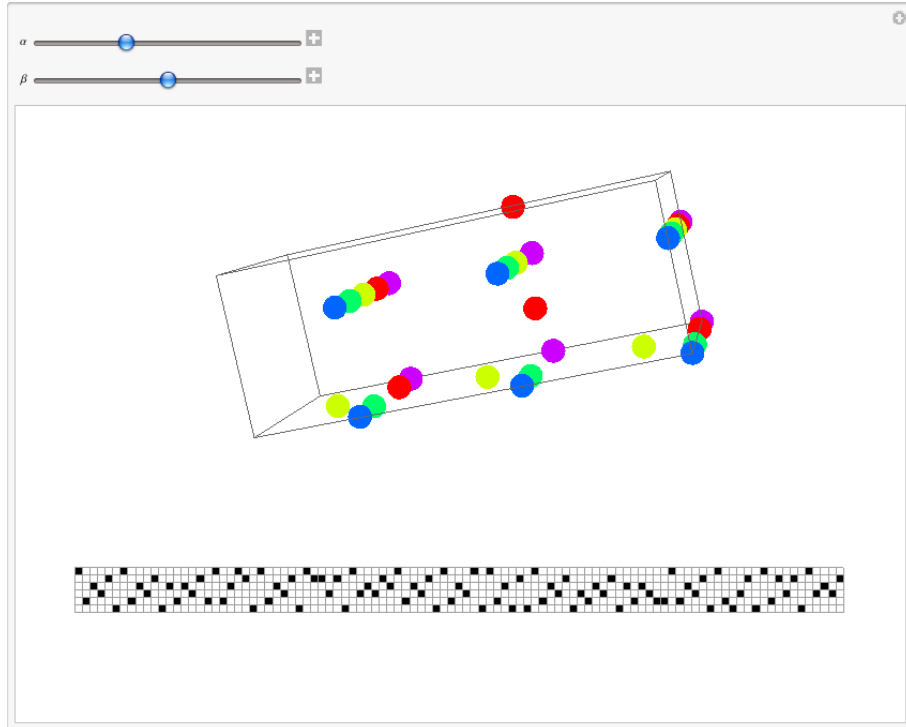


FIGURE 12. Perturbed lattice in 3D and the isomorphic Vuza canon.

²⁵I like to describe B as the banknotes, and A as the loose change.

²⁶With the understanding that opposite sides coincide, so it is a torus really.

To sum up, the algorithm is:

- Compute A, B as lists of triplets. It may be convenient to
 - (1) Keep a tensor structure (a matrix of triplets), and even
 - (2) keep B as *two* parts – the ‘lower triplets’ $(3x, 5y, 0)$ B_0 and the ‘upper’ ones $(3x, 5y, 1)$ in B_1 .
- Shift one (or several) columns of B_1 , say adding $(1, 0, 0)$ (or some multiple) to all elements of one column.
- Similarly shift a row (or several) in B_2 .
- Apply the isomorphism $(x, y, z) \mapsto \alpha x + \beta y + \gamma z$ from $\mathbb{Z}_{3a} \times \mathbb{Z}_{5b} \times \mathbb{Z}_2$ to $\mathbb{Z}_n$. Coefficients α, β, γ can be computed from the inverse isomorphism, which gives a system of congruences:

$$\begin{cases} \alpha x + \beta y + \gamma z \equiv x & \text{mod } 3a \\ \alpha x + \beta y + \gamma z \equiv y & \text{mod } 5b \\ \alpha x + \beta y + \gamma z \equiv z & \text{mod } 2 \end{cases}$$

so that for instance (taking $(x, y, z) = (0, 0, 1)$), γ is the multiplicative inverse of $15ab$ modulo 2, times $15ab$, so it is always $15ab$ since a, b are odd.

2.4. From canons to matrixes. A nice way to represent (mosaic) tilings is building matrixes for both inner and outer voices. In a more general context [6], it is shown how the convolution product of characteristic functions is isomorphic to the ordinary product of matrixes. The trick is to turn a set first into its characteristic function, which can be represented as a (column) vector, then to put side by side all circular permutations of this vector in order to get a *circulating matrix*.

For instance the tango rythm $\{0, 3, 4, 6\} \subset \mathbb{Z}_8$ turns into

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

If we denote by $M(A), M(B)$ the matrixes associated with subsets A, B , the important formula is

$$M(A \oplus B) = M(A) \times M(B) \quad (\mathcal{M})$$

Hence we have a mosaic canon whenever $M(A) \times M(B) = \mathbf{1}_n$, the matrix with ones everywhere. This does not allow an easier recipe for *solving* the tiling problem (i.e. finding all $M(B)$ ’s with 0 or 1 coefficients satisfying $(\mathcal{M})$ for a given $M(A)$), but some matricial techniques enable to provide at least one solution. Since these techniques are well implemented in many programming environments and work for somewhat large matrixes, this can be of interest to composers. One such technique is *linear programming*: since we want $M(A) \times b = \mathbf{1} \in \mathbb{R}^n$ (it is enough to test the product on the first column b of $M(B)$), we can try to *minimize* the linear form $b \mapsto b_1 + \dots b_n$ (a.k.a. $\#B$ when B is a true set) in the domain defined by inequalities $b \geq 0$ (or even $0 \leq b \leq 1$) and $M(A) \times b \geq 0$. Linear programming does that by following edges of the polytope thus defined, always choosing the edge that leads to the lowest value of the linear form.

Perhaps surprisingly, this almost always works.²⁷ Moreover, since the geometric (simplex) algorithm involved in linear programming is ‘looking for corners’, jumping from vertex to vertex, it tends to give packed solutions, which are usually non periodic: a good way to generate Vuza canons with large periods (from a few hundreds to 1,000 or so) is to start, as above, with a S_A , its Coven-Meyerowitz complement

²⁷An exception, running the native **LinearProgramming** routine in Mathematica 6.1, is (starting with $S_A = \{3, 25, 125\}, n = 750$), the tile $\{0, 87, 171, 213, 234, 375, 462, 546, 588, 609\}$. However, it is already quite convenient, though mysterious, that almost all solutions are in 0’s and 1’s.

(usually a very regular tile B_0) and then to iterate linear programming, finding $A_1, B_1, A_2, B_2, \dots$ until it loops (or, quite exceptionally, fails). Then $A_i \oplus B_j$ is usually a Vuza canon.

This kind of ping-pong game has been tried for all periods up to 1,000, in the (so far vain) hope of providing a canon where (T_2) would not be satisfied.

3. TILINGS WITH AUGMENTATIONS

We have a constellation of small results for this type of canons, which is of course of paramount historical importance since augmentation was a favorite compositional device for several not unconsiderable musicians.

However, this remains by and large *terra incognita*: for instance, it is still unknown whether it is possible to tile with a finite number of augmentations of any given (finite) tile, though we have the obvious

Theorem 16. *It is possible to tile the non negative integers $\mathbb{N}$ (and a fortiori $\mathbb{Z}$, tiling it in two halves) with augmentations of any finite given motif $A \subset \mathbb{Z}$.*

The trick is, if you have already covered a subset ALREADY of $\mathbb{N}$, to add whatever augmentation of A , offset to the first gap to the right, does not overlap ALREADY, and iterate. This greedy (and rather stupid) algorithm provides a tiling of $\mathbb{N}$, which is usually far from optimal. See below for a better fitting. Fig. 13 shows the beginning of this algorithm for $\{0, 1, 3\}$, where the successive augmentation ratios eventually cycle on the sequence $1, 2, 4, 4, 1, 2, 4, 4, \dots$

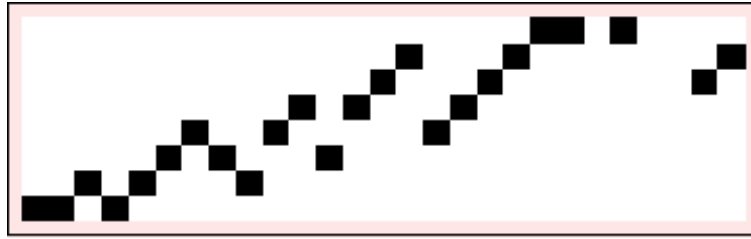


FIGURE 13. Tiling greedily with augmentations of $\{0, 1, 3\}$.

3.1. Johnson's perfect tilings. One of Tom Johnson's numerous ideas is trying to tile with augmentations of the simple tile $\{0, 1, 2\}$ with different ratios $i_1, \dots, i_k$. In terms of polynomials, it means finding a decomposition

$$1 + X + X^2 + \dots X^{n-1} = X^{n_1}(1 + X^{i_1} + X^{2i_1}) + X^{n_2}(1 + X^{i_2} + X^{2i_2}) + \dots X^{n_k}(1 + X^{i_k} + X^{2i_k})$$

with different i_k 's. In the general case, using $X = e^{2i\pi/3}$ in the last equation, it can be shown that the number of ratios divisible by 3 is a multiple of 3, and similar results, but little more is known theoretically about building solutions.

When the sequence of ratios is just $1, 2, \dots, k$, this problem is equivalent with that of Langford sequences. This has been explored computationally. The state of the art about perfect tilings can be found in this issue of *PNM* with J.P. Davalan's paper [13].

3.2. Johnson's JIM problem and generalizations. In 2001 Tom Johnson proposed building rhythmic canons by augmentation of motif $\{0, 1, 4\}$ with ratios of $2, 4, 8, \dots$ (it is impossible to tile with translations only) [21]. The smallest solution is on fig. 14.

Andranik Tangian devised a computer program, finding all solutions up to some given length [34]. It so happened that the length of all solutions was a multiple of 15. Variants of this problem exhibit similar behaviour. Surprisingly, the reason for this experimental result involves Galois theory in finite fields. Since in any field with characteristic p we have the Frobenius automorphism²⁸ $x \mapsto \mathcal{F}(x) = x^p$, we get

²⁸Meaning $\mathcal{F}(x + y) = \mathcal{F}(x) + \mathcal{F}(y)$ and $\mathcal{F}(x \times y) = \mathcal{F}(x) \times \mathcal{F}(y)$.

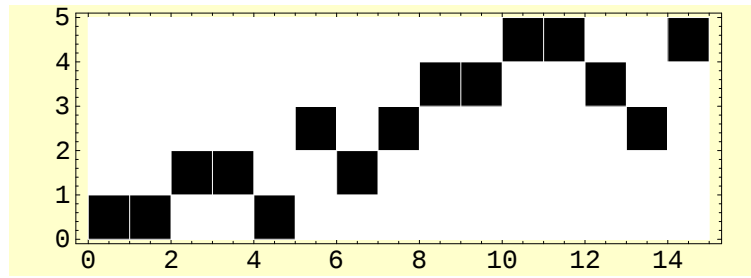


FIGURE 14. Smallest tiling with augmentations of $\{0, 1, 4\}$.

Lemma 2. *For any $P \in \mathbb{F}[X]$ where $\mathbb{F}$ is a finite field with characteristic 2, $P(X^{2^m}) = P(X)^{2^m}$.*

Now the polynomial equation expressing a tiling with augmentations (of ratio 2, 4, ...) of a motif A is (cf. [34])

$$A(X) \times B_0(X) + A(X^2) \times B_1(X) + \dots A(X^{2^m})B_m(X) = 1 + X + X^2 + \dots X^{n-1} = \frac{X^n - 1}{X - 1}$$

where the B_i 's are 0-1 polynomials indicating the different entries of augmented motif $2^i A$.²⁹

Mixing all this, we get that in general $A(X) \times (X - 1)$ must be a divisor of $X^n - 1$ (this is always possible for some n as shown by Theorem 20 below, but here we are looking for necessary conditions on n).

In the case of Johnson's problem, since $1 + X + X^4 = A(X)$ is irreducible in $\mathbb{F}_2[X]$, the quotient field $\mathbb{F}_2[X]/(A(X))$ is isomorphic with $\mathbb{F}_{2^4} = \mathbb{F}_{16}$, which shows that the roots of $A(X)$ in this appropriate extension are of order $15 = 16 - 1$ (since they belong to the group $\mathbb{F}_{16}^*$ with 15 elements). So $A(\alpha) = 0 \iff \alpha^{15} = 1$, i.e. if $A(X) \mid X^n - 1$, then necessarily $15 \mid n$.

This line of thought can be used for predicting the size of solutions with any (finite) tile: it must be a multiple of the lcm of the $2^k - 1$ where k runs over the degrees of the irreducible factors of $A(X)$ in $\mathbb{F}_2[X]$, see [3] for more details. However,

- getting these irreducible factors modulo 2 requires advanced programming (or software);
- this computation does not work for general augmentations (like 'perfect tilings'), only with ratios in 2^k , and
- the condition on size is necessary; it is not known whether this is sufficient for a solution to exist.

3.3. Using orbits of affine maps: autosimilarity and tilings. Once one's eye is set on rhythmic canons, they seem to crop up everywhere. For instance, the study of Johnson's autosimilar melodies [21, 2] revealed a possible construction of some rhythmic canons by augmentation.

Definition 9. *Let $M = (m_0, m_1 \dots m_n = m_0, \dots)$ be a n -periodic melody (the m_k 's are notes, or rests, or other musical events beginning on beat k). We say that M is autosimilar under the affine map $x \mapsto ax + b \pmod n$ when*

$$\forall k \ m_{ak+b} = m_k$$

meaning that augmenting the melody with ratio a and offsetting by b yields the same melody.

Famous examples include the Alberti bass (autosimilar under any odd ratio), Beethoven's fifth symphony's famous four-note motif, or Glenn Miller's *In the Mood*. Notice that a should be coprime with n for the map to be invertible.

Tom Johnson's intensive use of this notion made desirable a thorough mathematical study. The useful point for us here is the following characterization:

²⁹With the example above, $B_1(X) = X^5$ and $B_0(X) = 1 + X^2 + X^8 + X^{10}$.

Proposition 9. *Melody M is autosimilar under $\varphi : x \mapsto ax + b \pmod n \iff$ the set of occurrences of any event m , i.e. $\{k / m_k = m\}$ is an orbit (or a reunion of orbits) of the map φ .*

The question of tiling vs. autosimilarity arose when Tom Johnson found $\{0, 1, 3, 7, 9\}$, which tiles $\mathbb{Z}_{20}$ by translation and is also autosimilar with ratio 3. We have only one very partial result in the way of autosimilar melodies that tile by translation, with very short tiles (two notes):

Theorem 17. *A melody autosimilar with ratio $a \neq 1$ and period $n > 4$ gives a tiling of $\mathbb{Z}_n$ by translations of a 2-tile iff $n = 4k$, $a = 1 + 2k$, $b = \pm k$ with k odd.*

Two-note motifs may look a little barren, but it may be possible to build up longer tiling motifs with unions of these tiles. On the other end, there is more to be said on tilings by augmentation. A simple but productive case is the following:

Proposition 10. *For n prime, the orbits of the map $x \mapsto ax$ provide a tiling of $\mathbb{Z}_n^*$.*

For instance for $n = 7$, $a = 2$ we find orbits $\{1, 2, 4\}$ and $\{3, 6, 12\} = \{3, 6, 5\}$ which tile $\{1, 2, 3, 4, 5, 6\}$. $0=7$ is left standing alone, since it is a one-note orbit, cf. fig. 15. Tom Johnson pointed out that such isolated notes can be played as rests, providing interesting relief in the melody.

In more general situations, orbits are not generally the same size, which complicates the matter. However, there is another recipe, where tiles are no longer orbits but quite the reverse:

Lemma 3. *Any affine map whose orbits share **the same length** enables to build tilings with augmentation.*

Proof. Consider any set X transverse with the orbits, i.e. containing one point and only one from each orbit. Then $X, f(X), \dots, f^{r-1}(X)$ partition $\mathbb{Z}_n$, i.e. X tiles with its augmentations $aX + b$ a.s.o. $\square$

Example 6. *All the orbits of $f : x \mapsto 13x + 3 \pmod{20}$ have length 4:*

$\{0, 2, 3, 9\}, \{1, 6, 11, 16\}, \{4, 15, 17, 18\}, \{5, 7, 8, 14\}$ and $\{10, 12, 13, 19\}$. Take for instance the first elements: $X = \{0, 1, 4, 5, 10\}$. Applying f yields all following members of each orbit: $f(X) = \{3, 16, 15, 8, 13\}$. Iteration of the process gives a mosaic, where all motifs are images of the preceding one by the map f . Notice that one can choose any starting element in each orbit, thus finding 128 different tilings.

In general, when orbits have unequal lengths, this may yield a covering not a mosaic.



FIGURE 15. A tiling (packing) of $\mathbb{Z}_7$ with $\{1, 2, 4\}$ and augmentations.

Unfortunately, there is no known simple characterization of such affine maps. A partial discussion of this situation is given in the Online Supplementary of [2], wherein it is proved that

Lemma 4. *All orbits have the same length whenever the **smallest** orbit has length (some multiple of) $o(a)$, the order of a in the multiplicative group $\mathbb{Z}_n^*$.*

On the bright side, these constructions can be used in dimension 2 or more, providing tessellations of the torus $\mathbb{Z}_n^2$ and thereby tiling simultaneously pitch and time-span, for instance. On fig. 16 one can

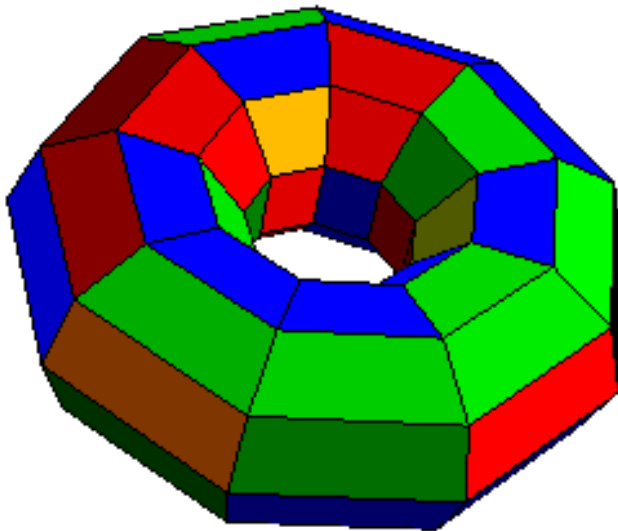


FIGURE 16. A 2D-tiling with “augmentations”.

see the orbits of matrix $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \pmod{9}$, tessellating the torus $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Each orbit is the image of another (the origin $(0, 0)$ excepted) by some matrix commuting with A .

In order to turn this method into a practical recipe for tiling, it would be necessary to predict which maps display orbits of equal length.

3.4. Find'em all algorithm. For many applications, there is nothing better to do than an exhaustive search by tree parsing, for instance when trying to find all mosaics with a given set of motifs. The general algorithm is as follows, it is generally easier to implement it recursively:

- Consider a set of motifs $M_1, \dots, M_k$ (all beginning with 0), and a set to tile (could be a multiset, or a cyclic group).
- Define recursively a procedure **FillIn** whose variables are: what part of the whole set is already covered (ALREADY), and how (SOFAR) it was filled, e.g., the list (or array) of the different translates of the different motifs used so far to fill ALREADY.
- If ALREADY is the whole set to tile, then add SOFAR to the set of solutions; else
- Compute FG, the first gap remaining to be filled (the smallest element of the complement set of ALREADY).
- For all motifs $M_1, \dots, M_k$ do
 - Compute $\text{FG} + M_i$.
 - Check if it fits, i.e. whether $(\text{FG} + M_i) \cap \text{ALREADY} = \emptyset$ (check also whether $\text{FG} + M_i$ is below the upper limit, when there is one). If not, exit; if it does, then
 - Add $\text{FG} + M_i$ to SOFAR, compute accordingly the new value of ALREADY and call **FillIn** recursively with those new values.

This algorithm is in exponential time, but guarantees all possible solutions. Depending on the precise tiling problem involved, some shortcuts may be found.

Several researchers have independently implemented their own version of this, which is basically a parsing of a tree whose nodes are the remaining gaps. The algorithm (in Fortran) is exposed in lavish detail in [35] for the case of $\{0, 1, 4\}$ and its augmentations; Jon Wild [43] had written and optimized a similar one for general purpose (in C), which is the quickest version so far; in high-level languages like Mathematica, it takes about fifteen lines of code and runs quickly enough for comparatively small

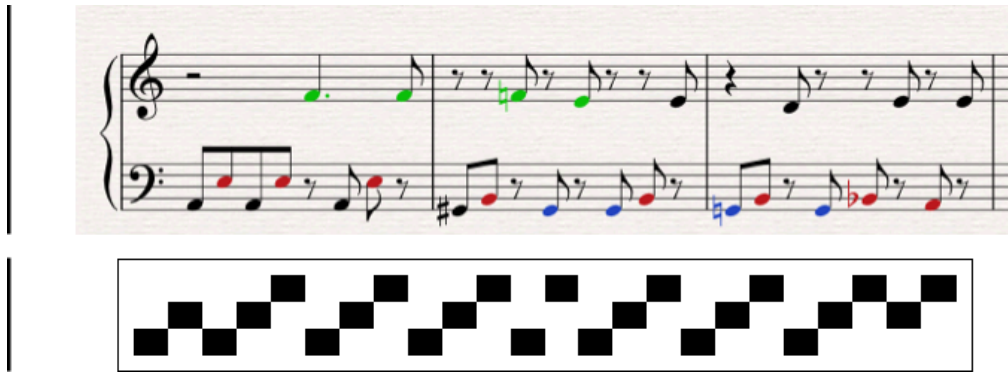


FIGURE 18. Tiling with a tango rhythm.

Actually this means that we get a *covering* (not a mosaic): the number of notes on each beat is not equal, but *congruent* to 1 (mod p), see beats 6, 9 and 10 where the number of notes is 3. So a musical score might look like figure 19.

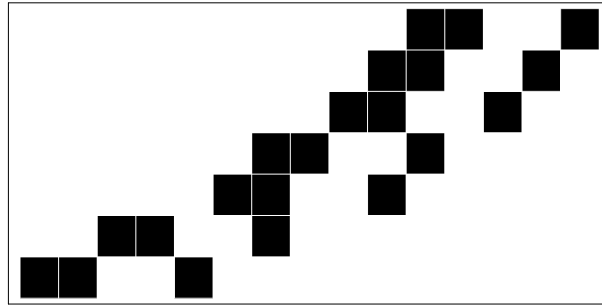


FIGURE 19. A tiling (covering) with an odd number of notes on each beat.

The proof involves non trivial Galois theory, see [3](Thm. 30). It hinges on

Theorem 20. *For any polynomial $P \in \mathbb{F}[x]$ where $\mathbb{F}$ is a finite field, such that $P(0) \neq 0$, P is a divisor of $X^n - 1$, for some $n \in \mathbb{N}$.*

According to [41], this was implicit in Galois's original papers.

Essentially, every root α of $A(X) \in \mathbb{F}_p[X]$ lives in some finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$, with $q = p^m$. Hence (by Lagrange's theorem) $\alpha^{q-1} = 1$ in $\mathbb{F}_q$, meaning that $X - \alpha$ is a factor of $X^{q-1} - 1$. With some twiddling (because of multiple roots) we get some large n (a multiple of all those $q - 1$ for all roots) such that $A(X)$ divides $X^n - 1$ – but this happens now in $\mathbb{F}_p[X]$ because all coefficients lie in that prime field. Applying this reasoning/algorithm to $A(X) \times (X - 1)$ instead, one gets an exact quotient $B(X) = \frac{X^n - 1}{(X - 1)A(X)}$, i.e. (T_0) is true. It only remains to turn $B(X)$ into a 0-1 polynomial³⁰ by the transformation rule (while $\alpha \geq 2$)

$$\alpha X^k \rightarrow (\alpha - 1)X^k + X^{n+k+1}$$

This is a constructive algorithm, though it is helpful to use some high-level software for computations in finite fields – especially factorization of the polynomial in irreducible factors in $\mathbb{F}_p[X]$). A simpler, but more cumbersome algorithm, would just test whether n is suitable, for larger and larger values of n of the form $n = p^j \text{lcm}_{1 \leq m \leq k}(p^m - 1)$ (the factor p^j allows for multiple roots):

```

j =0; m=0;
UNTIL A(X) divides X^n - 1 mod p
  DO j++; m++; n = p^j * lcm(p-1,p^2-1, ... p^m-1)

```

³⁰There is nothing more to be done when $p = 2$, of course, and hence any motif tiles a range $[0, n - 1]$ modulo 2.

(polynomial division modulo p is quite easy to implement even when not native to the programming environment).

In any event, it is better, when such an n is found, to check whether some of its divisors satisfies $(T_0) \bmod p$ for the given $A(X)$, in which (frequent) case a smaller solution exists.

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REFERENCES

- [1] Amiot, E., *Why Rhythmic Canons are Interesting*, in: E. Lluís-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, 190–209, Universität Osnabrück, 2004.
- [2] Amiot, E., *Autosimilar Melodies*, *Journal of Mathematics and Music*, July, vol. 2, n° 3, 2008, 157–180.
- [3] Amiot, E., *Rhythmic canons and Galois theory*, *Grazer Math. Ber.*, 347, 2005, 1–25.
- [4] Amiot, E., *À propos des canons rythmiques*, *Gazette des Mathématiciens*, SMF Ed., 106, 2005, 43–67.
- [5] Amiot, E., *New perspectives on rhythmic canons and the spectral conjecture*, in Special Issue “Tiling Problems in Music”, *Journal of Mathematics and Music*, July, vol. 3, n° 2, 2009.
- [6] Amiot, E., Sethares, W., *An Algebra for Periodic Rhythms and Scales*, Springer, 2040.
- [7] Andreatta, M., *On group-theoretical methods applied to music: some compositional and implementational aspects*, in: E. Lluís-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, 122–162, Universität Osnabrück, 2004.
- [8] Andreatta, M., Agon, C., (guest eds), *Special Issue “Tiling Problems in Music”*, *Journal of Mathematics and Music*, July, vol. 3, n° 2, 2009.
- [9] Andreatta, M., *De la conjecture de Minkowski aux canons rythmiques mosaïques*, *L'Ouvvert*, n° 114, p. 51–61, march 2007.
- [10] Chemillier, M., *Les Mathématiques naturelles*, chap. 5, Odile Jacob Ed., Pris, 2004.
- [11] Coven, E., and Meyerowitz, A. *Tiling the integers with one finite set*, in: *J. Alg.* (212), 1999, 161–174.
- [12] Agon, C., Amiot, E., Andreatta, M., *Tiling the line with polynomials*, *Proceedings ICMC 2005*.
- [13] Davalan, J.P., *Perfect rhythmic tilings*, *PNM*, 2011.
- [14] DeBruijn, N.G., *On Number Systems*, *Nieuw. Arch. Wisk.* (3) 4, 1956, 15–17.
- [15] Fidanza, G., *Canoni ritmici*, tesa di Laurea, U. Pisa, 2008.
- [16] Friertinger, H. *Remarks on Rhythmical Canons*, *Grazer Math. Ber.*, 347, 2005, 55–68.
- [17] Friertinger, H. *Tiling problems in music theory*, in: E. Lluís-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, Universität Osnabrück, 2004, 149–164.
- [18] Gilbert, E., *Polynômes cyclotomiques, canons mosaïques et rythmes k-asymétriques*, mémoire de Master ATIAM, Ircam, may 2007.
- [19] Hajós, G., *Sur les factorisations des groupes abéliens*, in: *Casopsis Pest. Mat. Fys.* (74), 1954, 157–162.
- [20] Hall, R., Klinsberg, P., *Asymmetric Rhythms and Tiling Canons*, *American Mathematical Monthly*, Volume 113, Number 10, December 2006, 887–896.
- [21] Johnson, T., *Tiling The Line*, proceedings of J.I.M., Royan, 2001.
- [22] Jedrzejewski, F., *A simple way to compute Vuza canons*, MaMuX seminar, January 2004, <http://www.ircam.fr/equipes/repmus/mamux/>.

- [23] Kolountzakis, M. *Translational Tilings of the Integers with Long Periods*, Elec. J. of Combinatorics **10**(1), R22, 2003.
- [24] Kolountzakis, M. & Matolcsi, M., *Complex Hadamard Matrices and the spectral set conjecture*, <http://arxiv.org/abs/math.CA/0411512>.
- [25] Kolountzakis, M. & Matolcsi, M., *Algorithms for translational tiling*, in Special Issue “Tiling Problems in Music”, Journal of Mathematics and Music, July, vol. 3, n° 2, 2009.
- [26] Laba, I., *The spectral set conjecture and multiplicative properties of roots of polynomials*, J. London Math. Soc. 65, 2002, 661-671.
- [27] Laba, I., and Konyagin, S., *Spectra of certain types of polynomials and tiling of integers with translates of finite sets*, J. Num. Th. 103, 2003, no. 2, 267-280.
- [28] Lagarias, J., and Wang, Y. *Tiling the line with translates of one tile*, in: *Inv. Math.* (124), 1996, 341-365.
- [29] Newman, D.J., *Tesselation of Integers*, J. Numb. Theory **9**, 1977., 107-11
- [30] Rahn, J., *Basic Atonal Theory*, New York, Longman, 1980.
- [31] Steinberger, J.P., *Tilings of the integers can have superpolynomial periods*, Combinatorica, **29**, 2009, 503-509.
- [32] Steinberger, J.P., *Multiple tilings of $\mathbb{Z}$ with long periods, and tiles with many-generated level semigroups*, New York Journal of Mathematics, **11**, 2005, 445-456.
- [33] Swenson, C., *Direct sum subset decompositions of $\mathbb{Z}$* , Pacific J. Math. **53**, 1974, 629-633.
- [34] Tangian, A., *The Sieve of Eratosthene for Diophantine Equations in Integer Polynomials and Johnson’s problem*, disc. paper N° 309, Fern Universität Hagen.
- [35] Tangian, A., *Constructing Rhythmic Canons*, PNM, Vol. 41, no. 2 Summer 2003, 66–95.
- [36] Tijdeman, R., *Decomposition of the Integers as a direct sum of two subsets*, in: *Séminaire de théorie des nombres de Paris*, 3D, Cambridge U.P, 1995, 261-276.
- [37] Sands, A.D., *The Factorization of abelian groups*, Quart. J. Math. Oxford, 10(2), 45–54.
- [38] Szabó, S., *A type of factorization of finite abelian groups*, Discrete Math. 54, 1985, 121–124.
- [39] Tao, T., *Fuglede’s conjecture is false in 5 and higher dimensions*, <http://arxiv.org/abs/math.CO/0306134>.
- [40] Vuza, D.T., *Supplementary Sets and Regular Complementary Unending Canons*, in four parts in: *Canons. Persp. of New Music*, nos 29(2) pp.22-49; 30(1), pp. 184-207; 30(2), pp. 102-125; 31(1), pp. 270-305, 1991-1992.
- [41] Warusfel, *Structures Algébriques finies*, Classiques Hachette, 1971.
- [42] Wild, J., *Tessellating the chromatic*, Perspectives of New Music, 2002.
- [43] Wild, J., *His new paper*, Perspectives of New Music, 2011.