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STURMIAN SEQUENCES AND MORPHISMS
A MUSIC-THEORETICAL APPLICATION

by

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Abstract. — Many properties of the mechanical word $s = (\lfloor (n+1)g \rfloor - \lfloor ng \rfloor)_{n=0,1,2,...}$ play a strikingly illuminating role in music theory when $g = log_2(3/2)$. This number represents the pitch height ratio between the musical intervals fifth and octave. The word s exemplifies the discrete rendering of the kernel of the linear pitch height form $p : \mathbb{Z}^2 \to \mathbb{R}$ with $p(m, n) = gm + n$ on the Pythagorean tone lattice, which is the free $\mathbb{Z}$-module of all linear combinations of these two intervals. Although this kernel as such should be musically redundant it turns out that the Christoffel-prefixes of its discrete rendering correspond to prominent musical tone systems ? as a side effect of the discretisation of a single pitch height level. More than that: Christoffel-duality and its extension to conjugates of Christoffel words provides deep insights into the constitution of musical modes. Sturmian morphisms provide a transformational meta-language to this study. It is particularly challenging to music-theoretically interpret the role of special standard morphisms within the monoid St_0 of all special Sturmian morphisms. For example, the authentically divided Ionian mode $(aaba, aab)$ is the image of the pair (a, b) under the special standard morphism GGD, sending a to $aaba$ and b to aab. The associated fifth up / fourth down - folding $(yx, yxyxy)$ from Fa to $Fa_\sharp$ is the image of the "reversed" special standard morphism DGG, sending x to yx and y to $yxyxy$. The theory of Sturmian sequences and morphisms is a rich treasure of mathematical facts of ideas with cross-connections to combinatorics, number theory, braid groups, symbolic dynamics and other domains, and all of which may potentially constribute to the music-theoretical interpretation of the apparently "redundant" pitch height kernel and its translates.

Contents

This presentation is an introduction to ongoing joint research with David Clampitt ([**15**]) and Manuel Dominguez ([**25**], [**35**], [**26**]), and Karst de Jong ([**23**], [**24**]). The first section introduces into the subject and traces some important findings which motivate the actual research. Section 2 portraits a study of *diatonic modes* by means of Christoffel words and their conjugates. Section 3 presents a transformational level of description in terms of Sturmian Morphisms, and Section 4 positions these contents with respect to a broader scope of music-theoretical questions.

1. Unsearchable Diatonic Scale

The seven-tone-diatonic scale $Do - Re - Mi - Fa - Sol - La - Si$ or $C - D - E - F - G - A - B$ is one of the most robust manifestations of organized musical material. Cuneiform sources support the assumption that the music of ancient mesopotamia already was based on a system of diatonic scales. European musical tradition is essentially based on this scale in quite different ways, such that it remains an open theoretical question wether these very different manifestations of the apparently 'same' scale will eventually find their place under one solid theoretical roof. It must be noted that in the last decades this question was regarded more an outmoded rather than an urgent problem. And even less so, the call for applied mathematics was mainly uttered by the mathe-musical enthusiasts alone. The numerous attempts to gain insight into tone relations through juggling with integer ratios acquired by and by the connotation of escapism from musical reality, especially in the second half of the 20th century. Innovative mathematical approaches to these outmoded questions still have therefore to deliberately dismiss the heritage of this connotation. Why should a sophisticated theory be needed to understand the interplay of seven tones? Curiously, fresh impulses came up from an unexpected direction, where some mathematical ideas entered the realm of music theory: atonal theory.

At a time when music theorists were engaged in the classification of all kinds of possible chords that can be formed with selections from the 12 tones per octave they looked also at the old-fashioned diatonic collection

$$Dia = \{0, 1, 3, 5, 6, 8, 10\} \subset \mathbb{Z}_{12}$$

and its 12 translates as instances of the $2^{12} = 4096$ subsets of the chromatic 12-tone system. In this chromatic system one considers the twelve tones as homogeneously distributed in the octave. The seven-note-subset Dia forms a specific inhomogenous subset, where some tones are directly neighboring each other (such as 0 and 1 or 5 and 6), while other successive tones are 2 chromatic units apart from each other, namely when there is a black tone between them (such as between 1 and 3, 3 and 5, 6 and 8, 8 and 10, 10 and $12 = 0$).[1] In contrast to this view, does the traditional arrangement of keys on a piano keyboard not support the homogeneity of the chromatic system. It is only the tuning in *equal temperament* which exemplifies the homogeneity of the tone space on the acoustic level. But with regard to key distribution one has to acknowledge that the seven white keys exemplify a homogenous distribution per octave, while the black keys are inserted as a secondary layer of keys. Traditional music notation also supports the idea of a homogenous distribution of the seven diatonic scale steps per octave through the graphical height of note heads on the staff. Recall, that each non-diatonic note shares the height position of a diatonic note and is then altered either by sharp- or flat-signs.

John Clough ([**16**]) therefore proposed a *generic* description of the diatonic scale in analogy to the approach of atonal theory (with 12 tones per octave) and classified all subsets of $\mathbb{Z}_7$ with respect to translation and inversion (multiplication by -1). The simultaneous investigation of the generic and the specific levels of description by Clough and Myerson ([**17**], [**18**]) marks a breakthrough in the study of the diatonic system.

1.1. Generic and Specific Levels of Description. — Figure 1 illustrates the basic observation, upon which the investigations of Clough and Myerson are built. It plots the tones of the diatonic scale as points in a two-dimensional space, which can be interpreted as $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}_{12} \times \mathbb{Z}_7$. The left drawing specifies this plot as the graph of the function: $f(t) = \lceil \frac{7}{12} t \rceil$ with $f : \mathbb{Z} \to \mathbb{Z}$ or $f : \mathbb{Z}_{12} \to \mathbb{Z}_7$. The right drawing displays the graph of the inverse function, namely $j(s) = f^{-1}(s) = \lfloor \frac{12}{7} s \rfloor$ with $j : \mathbb{Z} \to \mathbb{Z}$ or $j : \mathbb{Z}_7 \to \mathbb{Z}_{12}$.

In their investigation Clough and Myerson studied the general conditions, under which strictly analogous situations occur and explored several theoretical consequences. They investigated the general formula $j_{c,d}(s) = \lfloor \frac{c}{d} s \rfloor$ for coprime cardinalities c (number of chromatic tones) and $d < c$ (number of diatonic tones) and investigated the specific scale $j_{c,d}(0), ..., j_{c,d}(d-1) \in \mathbb{Z}_c$. For example, they pointed out that this scale has a *generalized circle of fifths*. This means, that – besides the map $j_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$ – there is another map $i_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$ which yields exactly the same specific scale, but in a permuted way. To that end, let $\iota_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$ denote the 'recasting' of the numbers $0, ..., d-1 \in \mathbb{Z}$ from their role as representatives for the residues mod d into their role as representatives for (the first d) residues mod c. If d^* denotes the multiplicative inverse of d mod c, the linear automorphism $\phi_{c,d} : \mathbb{Z}_c \to \mathbb{Z}_c$, sending $z \in \mathbb{Z}_d$ to $\phi(z) = (c - d^*)z$, maps the arithmetic sequence $\iota_{c,d}(\mathbb{Z}_d) = \{0, ..., d-1\} \subset \mathbb{Z}_c$ bijectively to the same set $j_{c,d}(\mathbb{Z}_d)$. In other words,

[1] The set *Dia* is the lexicographic first representative of its translation class. Identifying 0 with B, 1 with C, etc we obtain the 'white key' diatonic scale.

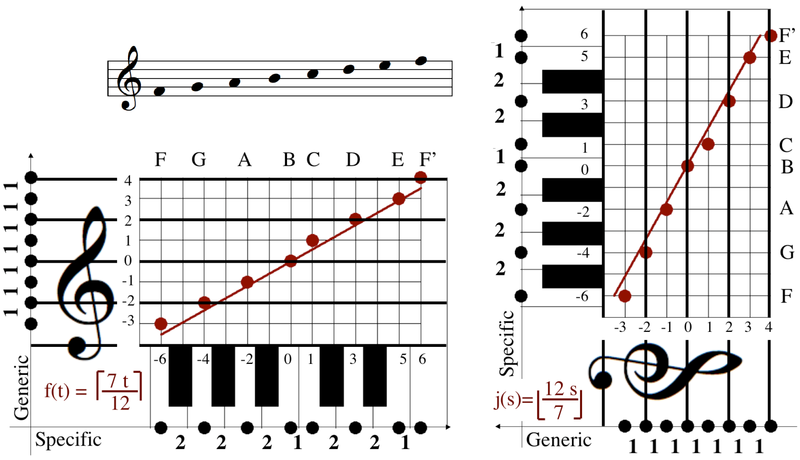

FIGURE 1. Left side: The generic level as a function of the specific level. The generic scale degree in $s = f(t) \in \mathbb{Z}$ (or $s \in \mathbb{Z}_7$) which is associated with the given specific chromatic pitch $t \in \mathbb{Z}$ (or chromatic pitch class $t \in \mathbb{Z}_{12}$) is the smallest integer $\lceil \frac{7}{12} t \rceil$ greater all equal to $\frac{7}{12} t$. Right side: The specific level as a function of the generic level. The specific scale degree $t = j(s) \in \mathbb{Z}$ (or $t \in \mathbb{Z}_{12}$) which is associated with the generic scale degree $s \in \mathbb{Z}$ (or $s \in \mathbb{Z}_7$) is the largest integer $\lfloor \frac{12}{7} s \rfloor$ smaller or equal to $\frac{7}{12} t$).

$i_{c,d} = \phi_{c,d} \circ \iota_{c,d} : \mathbb{Z}_d \to \mathbb{Z}_c$ is another way to create the tones of the scale, namely as a permuted arithmetic sequence. Figure 2 shows for the diatonic case, how this works.

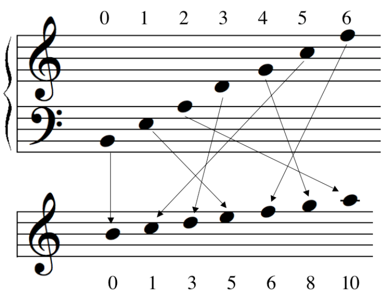

FIGURE 2. The diatonic scale is – up to octave displacement – a permuted sequence of fourths (or fifths).

The circumstance that the product $(c - d^*)d$ of the period $(c - d^*)$ of the arithmetic sequence and the cardinality d of the scale yields -1 has a music-theoretical important consequence, which is implicitly known to every musician, but which – in common understanding – is not appreciated as an extraordinary property of the diatonic scale.

Figure 3 shows that the fourth (or fifth-) transposition of a diatonic scale corresponds to the minimal alteration of one single tone. In search of microtonal alternatives to the prominent 12-tone system, Gerald Balzano ([**3**]) highlighted this property, among others.

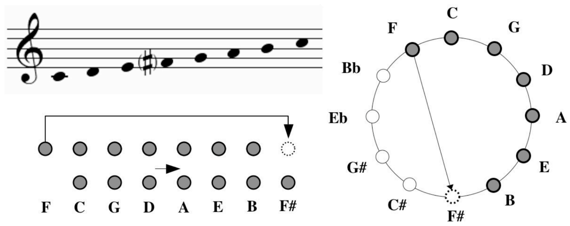

FIGURE 3. Two diatonic scales – C-major and G-major – in the same orbit under translation differ only in one single tone by a minimal distance, the chromatic alteration of the tone F to $F_\sharp$.

Clough and Myerson coined one of the consequences, which they proved for arbitrary co-prime cardinalities c and d, *Cardinality equals Variety.* This is closely related to the concept of *minimal complexity* in algebraic combinatorics on words (see Section 3 below). There are three types of triads in the diatonic scale: major triads (C-E-G, F-A-C, G-B-D), minor triads (D-F-A, E-G-B, A-C-E) and a diminished triad (B-D-F). Likewise, any fixed generic white-key-pattern of cardinality 3, comes in exactly three species, once it is shifted along the keyboard within $\mathbb{Z}_d$ and is classified with respect to translation in $\mathbb{Z}_c$. Figure 4 illustrates this principle for diatonic 3-tone-sets, but the concrete values $d = 7$ and $c = 12$ play no role.

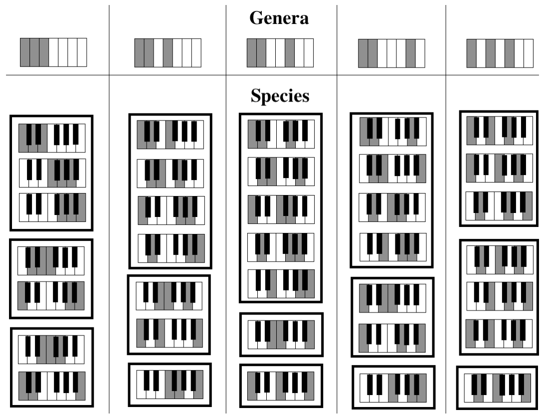

FIGURE 4. Cardinality equals Variety: the 35 3-tone chords fall into five translation classes in $\mathbb{Z}_7$ (genera). Under classification with respect to translation in $\mathbb{Z}_{12}$ each genus splits into three species.

The same principle explains, why in the diatonic scale there are exactly two types of seconds (major second M2, minor second m2), two types of thirds (major third M3, minor third m3), two types of fourths (pure fourth P4, augmented fourth A4), two types of fifths (pure fifth P5, diminished fifth D5), two types of sixths (major sixth M6, minor sixth m6), two types of seventh (major seventh M7, minor seventh m7).

The findings by Clough and Myerson assume that the diatonic scales are embedded into a homogenous chromatic ambient space $\mathbb{Z}_c$ and the musically relevant case is, of course, $c = 12$. One may wonder, wether the mathematical knowledge about analogous situations in other cardinalities is an enhancement of the music-theoretical knowledge proper. But the answer needs to be affirmative. Aside from the fact that musicians experiment with other tone systems, it is, of course, useful no know under which circumstances two apparently single facts are consequences of each other. This in itself is part of music-theoretical knowledge.

1.2. Well-formed Scales. — An axiomatic embedding of the diatonic scale into $\mathbb{Z}_{12}$ would be problematic from a music-theoretical point of view. Norman Carey and David Clampitt ([**10**], ([**11**]) succeeded to show that most of the above mentioned theory can be reformulated without making this assumption. The paradigmatic example for their approach is the *Pythagorean* diatonic scale, which is generated by the pitch height interval of a perfect fifth. It should be noted however, that the underlying controversy is not primarily concerned with norms for tuning keyboards. It is much more about the interdependence of music-theoretical concepts. Does any manifestation of a diatonic scale presuppose a manifestation of chromatism? Does any manifestation of chromatism presuppose a finite homogenous chromatic universe?[(2)] The theory of *well-formed scales* characterizes the conditions under which generated scales behave in a proper analogy to the Pythagorean diatonic scale.

Let the real numbers $\mathbb{R}$ denote a continuous pitch height (log-frequency) space, such that the integers $\mathbb{Z} \subset \mathbb{R}$ denote all those pitch heights which differ by multiples of octaves from a fixed pitch height 0. The factor group $\mathbb{R}/\mathbb{Z}$ denotes the continuous *chroma space* whose elements are pitch height classes under octave identification. The Pythagorean diatonic scale, as being generated by the pitch height interval $g = log_2(\frac{3}{2}) \in \mathbb{R}/\mathbb{Z}$ can be described as a map:

$$\gamma_g : \mathbb{Z}_7 \to \mathbb{R}/\mathbb{Z} \text{ with } \gamma_g(z) = zg \bmod 1 \text{ (for } z = 0, ..., 6 \in \mathbb{Z}).$$

The reordered image $\Sigma = \gamma_g(\mathbb{Z}_7)$ of γ_g with respect to the linear order of the real numbers in the fundamental domain $[0, 1)$ can be represented by another map:

$$\sigma_g : \mathbb{Z}_7 \to \mathbb{R}/\mathbb{Z} \text{ where } \sigma_g(\mathbb{Z}_7) = \gamma_g(\mathbb{Z}_7) \text{ and } \sigma_g(0) < \sigma_g(1) ... < \sigma_g(6).$$

(2) The dependence or independence of these questions from issues of tuning is a problem in its own right.

If we identify $\mathbb{Z}_{12}$ with the subgroup $\frac{1}{12}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ and consider the generator $g = \frac{5}{12}$, we see that the maps σ_g and δ_g coincide with the maps $j_{12,7}$ and $i_{12,7}$ of the previous subsection. But, while δ_g and $i_{12,7}$ are actually defined in the same way, σ_g is a rather unhandy bureaucratic 'paraphrase' of the definition for $j_{12,7}$. In order to know a single value for $j_{12,7}$ we calculate it directly, while we need to reorder the whole set $\delta_g(\mathbb{Z}_7)$ before we can determine a single value for the map σ. The desired loophole is provided by the permutation $\pi_g : \mathbb{Z}_7 \to \mathbb{Z}_7$ for which $\sigma_g = \gamma_g \circ \pi_g$. Carey and Clampitt observed that this permutation π_g behaves nicely whenever the scale does. More precisely, whenever π_g is a linear automorphism of $\mathbb{Z}_d$, the scale is either completely regular or it has the "cardinality equals variety" - property. Therefore, Carey and Clampitt ([**10**] call a generated scale $\sigma_g = \gamma_g \circ \pi_g : \mathbb{Z}_d \to \mathbb{R}/\mathbb{Z}$ *well-formed*, if the permutation π_g is a linear automorphism of $\mathbb{Z}_d$.

This means musical terms, that every specific instance of the generator interval (including the closing interval between the last generated tone (chroma point) $(d-1)g$ and the starting point 0 spans the same number of scale steps (no matter which size they have). Conversely, every instance of a step interval is divided into the same number of generator intervals. In the case of the fifth-generated diatonic scale every fifth (including the diminished fifth) comprises 4 steps and every step (minor second or major second) comprises to fifths. Händel's Passacaille for Harpsichord (see Figure 5) illustrates this in the way how the bass progresses at three different metrical levels.

FIGURE 5. Circular definition of the diatonic scale: a step is two fifths, a fifth is four steps. The Passacaille from Georg Friedrich Händel's Suite VII for Harpsichord exemplifies this musically. At bar level there are descending lines of three steps down: G - F - Eb - D. At half bar level the fundamental bass forms a diatonic circle of fifths: G - C - F - Bb - Eb - A - D - G (Note that although the real bass of the second chord is Eb, its fundamental bass is nevertheless C). In the first halves of bars 5, 6, 7, 8 the falling fifths are filled with descending four-step-lines at eighth-notes-level.

The variety of all well-formed scales is controlled by the Stern-Brocot-Tree. A g-generated scale of cardinality d is well-formed if and only if g has a semi-convergent $\frac{k}{d}$ with denominator d. The same linear automorphism $\pi_g : \mathbb{Z}_d \to \mathbb{Z}_d$ is shared by all generators g, whoose paths along the Stern-Brocot-tree share a prefix till some semi-convergent $\frac{k}{d}$ with denominator d.

The sequence of semi-convergents of a fixed generator g yields a hierarchy a well-formed scales. Figure 6 shows the fifth-generated well-formed scales which are associated with the semi-convergents $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$ and $\frac{7}{12}$. All of these scales are of musical relevance. According to the Cardinality-equals-Variety property there are two step species in each scale. Their difference is called the *comma* of this scale.(3) Any finite hierarchy of scales like this is stable under slight changes of the generating interval g.

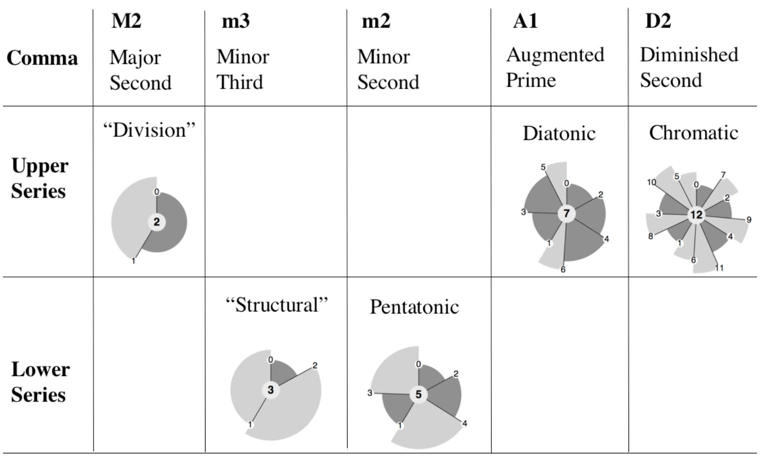

FIGURE 6. The hierarchy of fifth-generated well-formed scales. The drawings display the scales as arrangements of arc-segments around a circle. Their radii embody the generation order of the scale. The two different scale step intervals are distinguished by different gray levels.

Yves Hellegouarch ([**28**], ([**29**], ([**30**]) achieved very similar results by considering cyclic factor groups of the free group $\mathbb{Z}[g]$, generated by octave and fifth, divided by a comma-group. The fact, that the specific steps of a well-formed scale differ exactly by a comma, implies that the the group of comma classes is cyclic. Thus, in his approach, the generic level of a scale is realized as a factor group.

(3)The diminished second is known as *Pythagorean comma*, which is a rather small interval. A comma at level n of the hierarchy appears as a scale step at level $n+1$.

2. Well-Formed Modes

In this section a recent refinement of the generic level of description is presented (c.f. [**15**]), which applies facts about *Christoffel words* and their conjugates to the study of diatonic modes. The music-theoretical motivation for this refinement is the problem that in $\mathbb{R}/\mathbb{Z}$ the octave is completely ignored. But music is sensitive towards register such that there is a desire to reformulate the theory of well-formed scales without factoring the octaves completely out. Christoffel words and their conjugates provide a well adopted theoretical framework for this goal.(4)

The diatonic scale comprises in each octave register five whole steps C-D, D-E, F-G, G-A, A-B and two half steps E-F, B-C. In the sequel we use the two letter alphabet $\{a, b\}$ to denote binary scale step patterns as words, such as *aaabaab*, *aabaaab*, etc. In the diatonic case a stands for whole step and b for half step. Between two half steps there are two or three whole steps, respectively. The multiplicities 2 and 5 are co-prime and therefore the step distribution of the diatonic is the most regular one within words of length 7. However, the amount of irregularity which is exemplified by the distinct factors *aaa* and *aa* guaranties that all seven *diatonic modes* Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian and Locrian have individual step patters. This is yet another manifestation of the "Cardinality equals variety"-property, but in this case for sequences instead of sets. Each mode starts with another tone and each mode has a distinct step pattern. All seven patterns (see left column of Figure 8) comprise a full conjugacy class of 7-letter words.

In music theory one distinguishes two subdivisions of these scale patterns which are called *authentic division* and *plagal division.* In the authentic case the tone of reference – which in music theory is called *finalis* – is the lowest tone of the scale. The right column of Figure 8 displays the seven diatonic modes in a different way, which deserves a preparatory remark. In Section 1 we have seen that if one disregards octave register, the tones of the "white-key" diatonic scale can be obtained as an arithmetic sequence of starting from the tone F with a period of the interval of a fifth: F-C-G-D-A-E-B. But as this sequence does not stay within the register of one octave, is not the diatonic scale in the literal sense. The tones of that sequence can instead be folded into one octave register by replacing a fifth up by a fourth down, whenever the register would be transgressed. Figure 7 illustrates this for two cases.

All scale-foldings in the right column of Figure 8 are sensitive to the individual octave registers, where the finalis is the lowest note of the mode. These octave registers are contiguous fundamental domains for $\mathbb{Z}_7$ within the integers $\mathbb{Z}$, where we replace

(4) With a few exceptions [**12**],[**13**], musical scale theory and combinatorial word theory have remained unaware of each other, despite having an intersection in methods and results that by now is considerable. The theory of words has a long history, with many developments coming in the last few decades; see [**31**] for an account. I thank Franck Jedrzejewski for an initial reference in word theory and Valérie Berthé for helpful comments on [**34**].

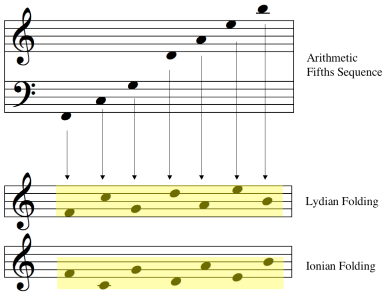

FIGURE 7. Folding of the arithmetic sequence of fifths into the octave registers of the Lydian and Ionian modes.

several instances of the upward-fifth by downward fourths. In the scale foldings we use the two letter alphabet $\{x, y\}$ with x denoting *fifth up* and y denoting *fourth down*.

Figure 8 shows the authentic division, where the octave is divided in a fifth (comprising four steps) and a fourth (comprising three steps). The dividing tone is traditionally called *confinalis*. The division is indicated by a vertical line: $aaba|aab$ and – to be more precise – the notation $u|v$ with $u, v \in \{a, b\}^*$ is an abbreviation for the word-triple (uv, u, v). In the plagal case the finalis divides the word into a fourth and a fifth, while the scale starts at the confinalis as its lowest tone.

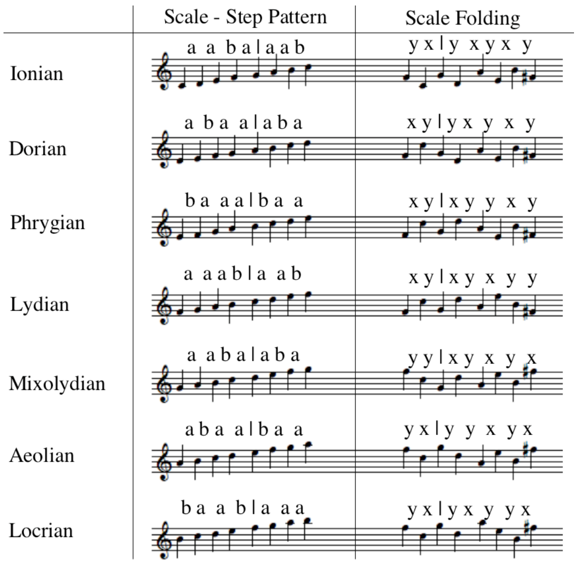

FIGURE 8. Scale-Step Patterns (whole step = a, half step = b) and Scale Foldings (fifth up = x, fourth down = y).

The scale foldings in the right column of Figure 8 have also a division into a prefix of length 2 followed by a suffix of length 5. This division becomes clear in Section 3, when Sturmian Morphisms are used to generate the modes. At first sight, one might be puzzled by the occurrence of the tone $F_\sharp$ is the last tone in each of the seven scale foldings. It is a "black key" and – strictly speaking – does not belong to the diatonic scale. Closer inspection shows that this eighth tone behaves in strict analogy to the octave of the finalis. This repetition of the finalis, one octave higher, can be regarded as closely related variant of the same essential scale degree, although being remote with respect to pitch hight. Analogously one may regard $F_\sharp$ as a close neighbor of F, i.e. as a variant of the same essential scale degree, although being remote with respect to tone kinship (measured in fifths). The interval between F and $F_\sharp$ is called *augmented prime*.

2.1. Christoffel Duality. — The following mathematical facts suggest a more tight connection between octave and augmented prime, rather than a mere analogy. The two columns of Figure 8 exemplify a refinement, of what is known as *Christoffel duality* within *algebraic combinatorics of words*. Here we simply recall a definition and further below remind of an equivalent characterization of Christoffel words (following the recent paper by Valérie Berthé et. al. [**5**] on Christoffel duality):

Definition 1. — *For two co-prime positive integers p and q the Christoffel word of slope $\frac{p}{q}$ over the ordered aphabet $\{a < b\}$ has length $n = p + q$ and has the form $w = w_1 \dots w_n$ with*

$$w_i = \begin{cases} a & \mathit{if} \quad p \cdot i \ \mathit{mod}\, n > p \cdot (i-1) \ \mathit{mod}\, n \\ b & \mathit{if} \quad p \cdot i \ \mathit{mod}\, n < p \cdot (i-1) \ \mathit{mod}\, n \end{cases}$$

To begin with, one may observe that the scale step pattern $aaabaab$ of the lydian mode is indeed the Christoffel word of slope $\frac{2}{5}$, and the folding $xyxyxyy$ is the Christoffel word of slope $\frac{4}{3}$ (over the ordered alphabet $\{x < y\}$). Thus, the scale-step pattern as well as the scale folding of the lydian mode are Christoffel words. The duality of both words can be expressed in several ways. For example, the numerators 2 and 4 of the slopes $\frac{2}{5}$ and $\frac{4}{3}$ are mutual inverses mod 7 and so are the the denominators 5 and 3.

A more advanced expression of the duality is based on the specific structure of Christoffel words. One can see immediately from the definition that any Christoffel word w over the ordered alphabet $\{a < b\}$ starts with a and ends with b. If we write $w = aub$, it is true (but less obvious), that the remaining central factor u of length $n-2$ is a doubly periodic palindrome with the two periods p^* and q^* (where $\frac{p^*}{q^*}$ is the slope of the dual word). Such palindromes are called central palindromes. It can be shown that they have a unique decomposition of the form $w = rabs$ with palindromic words r and s. In our example we have $aaabaab = aub$ with palindrome $u = aabaa$

having periods 3 and 4, and $xyxyxyy = xvy$ with palindrome $v = yxyxy$ having periods 2 and 5. The decomposition of u is $(a)ab(aa)$ and the decomposition of v is $(yxy)xy()$.

Such doubly periodic palindromes can be explicitly constructed as *right iterated palindromic closures* of suitable *directive words*. To that end one first defines the (simple) *right palindromic closure* of a word w as the unique smallest palindrome w^+ having w as a prefix. This means: one writes $w = uv$ with v being the largest palindromic suffix of w. If $\tilde{u}$ denotes the reversal of the word u one obtains $w^+ = uv\tilde{u}$. The right iterated palindromic closure $Pal(w)$ of a word $w = w'w_n$ is defined recursively as $(Pal(w')w_n)^+$, where w_n denotes the last letter of w. The Palindromic closure of the empty word ϵ is supposed to be ϵ itself.

In our examples the central palindrome u is of the form $u = Pal(aab)$ of the *directive word* $dir(u) = aab$, because $Pal(aab) = (((a^+)a)^+b)^+ = (aab)^+ = aabaa$. The central palindrome v is the *palindromic closure* $v = Pal(yxx)$ of the *directive word* $dir(v) = yxx$, because $Pal(yxx) = (((y^+)x)^+x)^+ = ((yx)^+x)^+ = (yxyx)^+ = yxyxy$.

The Christoffel duality between the two words $aaabaab$ and $xyxyxyy$ manifests itself in the fact that the two directive words $dir(u) = aab$ and $dir(v) = yxx$ are reverses of each other – up to respelling. Let $\psi : \{a, b\} \to \{x, y\}$ denote the order-preserving “respelling” of words in the two-letter-alphabet $\{a < b\}$ by words in $\{x < y\}$ with $\psi(a) = x$ and $\psi(b) = y$. yxx is the respelled reversal $\psi(\widetilde{aab}) = \psi(baa)$ of aab and thus the following equation connects the scale step pattern and the scale folding of the lydian authentic mode:

$$xvy = \psi(aPal(\widetilde{dir(u)})b).$$

The remaining 6 modes have scale-step patterns, which are conjugated to the lydian scale-step pattern $aaabaab$ and have scale foldings which are conjugated to the lydian scale folding $xyxyxyy$. Christoffel-Duality is usually understood as a duality between Christoffel words or between conjugacy classes thereof. But the clear music-theoretical meaning of Figure 8 imposes the desire for a refinement of Christoffel duality in terms of a pointwise bijection between the two conjugacy classes.

2.2. Refined Christoffel Duality.

— For every word $w \in \{x, y\}^*$ let $|w|_x$ and $|w|_y$ denote the multiplicities of the letters x and y in w, respectively. Let $|w|$ denote the length of w and for each $k \in \{1, ..., |w|\}$ let w_k denote the k-th letter of w.

Definition 2. — *Consider a two-letter word $w \in \{x, y\}^*$. The* balanced evaluation *of the alphabet $\{x, y\}$ with respect to w is the map $ev_w : \{x, y\} \to \mathbb{Z}$ with $ev_w(x) = |w|_y$ and $ev_w(y) = -|w|_x$. This induces a balanced evaluation of the word w, namely the map $\beta_w : \{1, \ldots, |w|\} \to \mathbb{Z}$ with $\beta_w(k) = ev_w(w_k)$. The* balanced accumulation *of w is the map $\alpha_w : \{0, \ldots, |w| - 1\} \to \mathbb{Z}$ of partial sums of the sequence $(\beta_w(1), \ldots \beta_w(|w| - 1)$, i.e. $\alpha_w(k) := \sum_{l=1}^{k} \beta_w(l)$.*

The attribute 'balanced' reflects that the total sum $\sum_{l=1}^{|w|} \beta_w(l) = 0$ over β_w yields zero. Its meaning differs from the concept of a *balanced word*.

If the word w of length n is conjugate to a Christoffel word, it turns out that the n values $\{\alpha_w(0), ..., \alpha_w(n-1)$ of its balanced accumulation form a contiguous fundamental domain for $\mathbb{Z}_n$ within $\mathbb{Z}$. This observation motivates the following definition and proposition.

Definition 3. — *A word w is called* well-formed *if there exists an integer $m_w \in \{0, ..., |w|-1\}$ such that $\{\alpha_w(0)+m_w, ..., \alpha_w(|w|-1)+m_w\} = \{0, ..., |w|-1\}$. $-m_w$ is called the* mode *of w. If m_w^* is the index for which $\alpha_w(m_w^*) = -m_w$ we call $-m_w^*$ the adjoint mode of w.*

Proposition 1. — *A word w is well-formed if and only if it is a Christoffel word or conjugate thereof. It is actually a Christoffel word iff its mode is zero: $m_w = 0$ (see* [**26**] *for a proof).*

In the sequel we use the term *well-formed word* interchangeably with *conjugate to a Christoffel word.*(5) On the basis of the balanced accumulation we may calculate the *plain adjoint* $w^\square$ of a given well-formed word w as follows. With the word w we associate a subset $X_w \subset \mathbb{Z}^2$:

$$X_w \quad = \quad \{(k - m_w^*, \alpha_w(k)) \,|\, k \in \{0, ..., |w|-1\}\}$$

The first coordinates vary within the contiguous fundamental domain

$$\{-m_w^*, ..., |w| - m_w^* - 1\}$$

for $\mathbb{Z}_{|w|}$, while the second coordinates vary within the contiguous fundamental domain

$$\{-m_w, ..., |w| - m_w - 1\}$$

for $\mathbb{Z}_{|w|}$. Furthermore, the reduction of this set X_w modulo $|w|$ yields the graph $\Gamma_f \subset \mathbb{Z}_{|w|} \times \mathbb{Z}_{|w|}$ of an affine automorphism $f : \mathbb{Z}_{|w|} \to \mathbb{Z}_{|w|}$, whose inverse f^{-1} results in the same manner from the plain adjoint $w^\square$. For some reason, to be discussed further below, we prefer to stay within $\mathbb{Z}\times\mathbb{Z}$ and define the plain adjoint as follows: If we order the set X_w with respect to increasing second coordinates and project the resulting sequence S_w to the first coordinate, we obtain a sequence $\pi_1(S_w)$ which – by definition of m_w^* – starts with 0. Thus there exists a unique well-formed word $w^\square$ of the same length $|w^\square| = |w|$ whose balanced accumulation sequence $(\alpha_{w^\square}(0), ..., \alpha_{w^\square}(|w|-1))$ coincides with the sequence $\pi_1(S_w)$. This word $w^\square$ is the plain adjoint of w.

Figure 9 sketches for the case of the Ionian mode how the plain adjoints of the scale-step pattern and the scale folding are calculated "by hand". For the sake of clear

(5)In mathematical music theory the attribute *well-formed* has been applied to those generated scales modulo octave, where the permutation from generation order into scale step order is a linear automorphism (c.f. [**10**]).

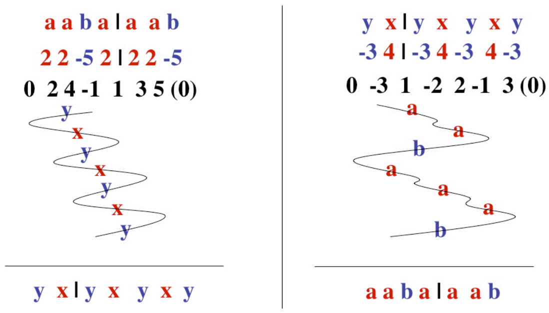

FIGURE 9. Left side: Calculation of the Ionian scale folding as the plain adjoint of the Ionian scale-step pattern. Right side: Calculation of the Ionian scale-step pattern as the plain adjoint of the Ionian scale folding.

distinction between the different music-theoretical meanings we switch the alphabet from $\{x < y\}$ to $\{a < b\}$ and vice versa.

At first sight it seems reasonable to simply identify the pair of conjugacy classes of Christoffel words with the associated pair of conjugacy classes of affine automorphisms of $\mathbb{Z}_{|w|}$ (conjugation with translations). For the Ionian scale folding $w = yxyxyxy$ we have the set

$$X_w = \{(-3,0), (-2,2), (-1,4), (0,-1), (1,1), (2,3), (3,5)\}$$

which reduces itself to

$$X_w = \{(4,0), (5,2), (6,4), (0,6), (1,1), (2,3), (3,5)\} \bmod 7$$

and is the graph Γ_f of the affine automorphism $f(z) = 2z + 6 \bmod 7$ with the inverse $f^{-1}(z) = 4z + 4 \bmod 7$. In the Lydian case we have the linear automorphisms $f(z) = 2z \bmod 7$ with the inverse $f^{-1}(z) = 4z \bmod 7$.

But from the music-theoretical point of view it seems to be important to be sensitive towards the distinction between the meanings of the letters x and y, namely fifth-up and fourth down. They can be expressed through the distinct 'vectors' $(1,4)$ and $(1,-3)$ in $\mathbb{Z} \times \mathbb{Z}$, which are conflated in $\mathbb{Z}_7 \times \mathbb{Z}_7$. Likewise, one wishes to distinguish between the meanings of a and b, namely whole-step and half step. They can be expressed through the distinct 'vectors' $(2,1)$ and $(-5,1)$ which are also conflated in $\mathbb{Z}_7 \times \mathbb{Z}_7$.

Figure 10 provides a geometric rendering of the tables in Figure 8, and thereby combines the discrete torus $\mathbb{Z}_7 \times \mathbb{Z}_7$ with the free $\mathbb{Z}$ - module $\mathbb{Z} \times \mathbb{Z}$ by simply attaching to each point in $\mathbb{Z}_7 \times \mathbb{Z}_7$ a copy of $\mathbb{Z} \times \mathbb{Z}$.

The cartesian product $(\mathbb{Z}_7 \times \mathbb{Z}_7) \times (\mathbb{Z} \times \mathbb{Z})$ serves as a discrete analogue to a tangent bundle. If one considers the scale-step patterns and scale foldings only as trajectories $\tau : \{0, \ldots, 7\} \to \mathbb{Z}_7 \times \mathbb{Z}_7$ there wouldn't be any distinction between affine lines and zigzags. If, however, we are given a vector field, whose 'integration' yields those

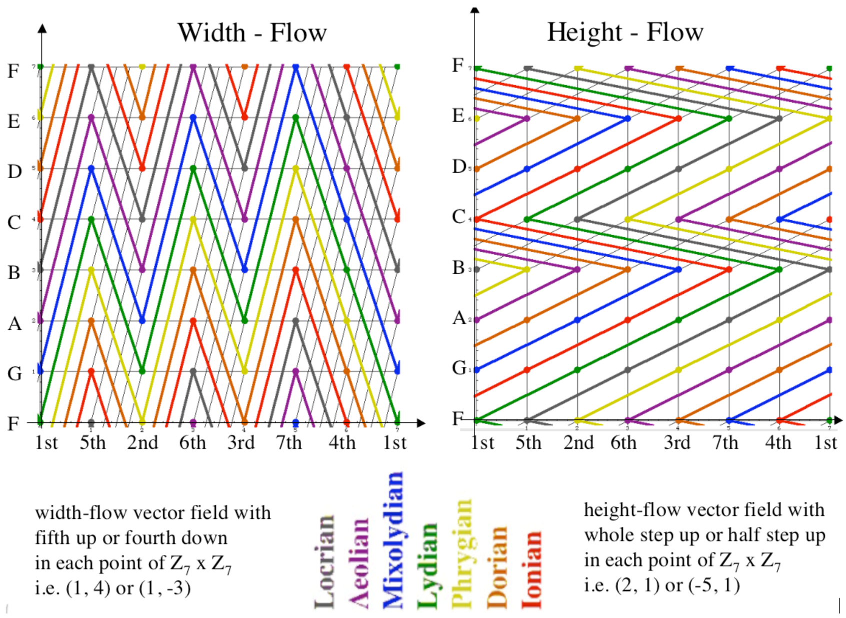

FIGURE 10. Scale foldings and Scale-step patterns as width- and height flows. The seven tones of each of the seven modes exhaust the 49 points in $\mathbb{Z}_7 \times \mathbb{Z}_7$.

trajectories, we may well distinguish between different types of flows. Augmented primes and Octaves can be balanced through winding numbers. See [**26**] for further investigations of Christoffel Duality in connection with tangent maps.

3. Sturmian Morphisms

Christoffel words and their conjugates have infinite analogous, which are called *Sturmian words* or *Sturmian sequences*. These words are sometimes considered one-sided infinite or two-sided infinite. In Section 3 we will study *the* music-theoretically central instances of such a word: the infinite *Pythagorean* word and its translates.

At this point we recall the definition of the Sturmian words in order to motivate the concept of *Sturmian Morphisms*, which preserve their structure of *minimal complexity*.

Definition 4. — *Let w denote an infinite word over the alphabet $\{0, 1\}$. For every natural number $n > 0$ let $Factors_n(w) \subset \{0, 1\}^*$ denote the set of finite words which occur as factors of length n within the infinite word w. The infinite word w is called a* Sturmian word, *if $Card(Factors_n(w)) = n + 1$ for every $n > 0$.*

Sturmian words can be explicitly constructed as *mechanical words* with irrational slope (c.f. [**31**], chapter 2). Given two real numbers α and ρ with $0 \leq \alpha \leq 1$ one defines the *lower* and *upper* mechanical words of *slope* α and *intercept* ρ as:

$$\begin{array}{rcl} \underline{s}_{\alpha,\rho}(n) & := & \lfloor (n+1)\alpha+\rho \rfloor - \lfloor n\alpha+\rho \rfloor \\ \bar{s}_{\alpha,\rho}(n) & := & \lceil (n+1)\alpha+\rho \rceil - \lceil n\alpha+\rho \rceil \end{array}$$

Every map $f : \{0,1\} \to \{0,1\}^*$ which associates finite words $f(0)$ and $f(1)$ with the letters 0 and 1 can be extended to an endomorphism of the free monoid $\{0,1\}^*$ of finite words and can further be extended to set of infinite words. Such an monoid endomorphism is called *Sturmian* if its extension to infinite words maps every sturmian word to a sturmian word. The monoid St of Sturmian morphisms contains a submonoid St_0 of *special Sturmian morphisms*[(6)], which is generated by the morphisms $G, \tilde{G}, D, \tilde{D} : \{x,y\}^* \to \{x,y\}^*$ with

$$\begin{array}{llll} G(x) = x, & G(y) = xy, & \tilde{G}(x) = x, & \tilde{G}(y) = yx \\ D(x) = yx, & D(y) = y, & \tilde{D}(x) = xy, & \tilde{D}(y) = y. \end{array}$$

If one adds the morphism E which exchanges the letters $E(0) = 1, E(1) = 0$ to the generators $G, \tilde{G}, D, \tilde{D}$, one obtains a list of generators for the full Sturmian monoid St. We use the same symbols $G, \tilde{G}, D, \tilde{D}, E$ to denote the corresponding morphisms even if we switch the (ordered) alphabet $\{0,1\}$ to $\{a,b\}$ or $\{x,y\}$.

3.1. Generation of the Diatonic Modes. — In each conjugacy class of a Christoffel word of length n (over the alphabet $\{a < b\}$) there are $n-1$ words which can be obtained as images $f(a|b) = f(a)|f(b)$ of the initial (divided) word $a|b$ under some special Sturmian morphism $f \in St_0$. One word is amorphous and is sometimes called the "bad conjugate". In the case of the scale-step patterns of the diatonic modes, it is the Locrian pattern $baab|aaa$, which is amorphous. This is in good accordance with the music-theoretical status of this mode. It is considered as a theoretical artifact with almost no practical relevance in music. Heinrich Glareanus rejects it from his system of 12 authentic and plagal modes.

The table below lists the 6 authentic modes:

Ionian	$GGD(a\|b)$	$=$	$GG(ba\|b)$	$=$	$G(aba\|ab)$	$=$	$aaba\|aab$
Dorian	$G\tilde{G}D(a\|b)$	$=$	$G\tilde{G}(ba\|b)$	$=$	$G(baa\|ba)$	$=$	$abaa\|aba$
Phrygian	$\tilde{G}\tilde{G}D(a\|b)$	$=$	$\tilde{G}\tilde{G}(ba\|b)$	$=$	$\tilde{G}(baa\|ba)$	$=$	$baaa\|baa$
Lydian	$GG\tilde{D}(a\|b)$	$=$	$GG(ab\|b)$	$=$	$G(aab\|ab)$	$=$	$aaab\|aab$
Mixolydian	$G\tilde{G}\tilde{D}(a\|b)$	$=$	$G\tilde{G}(ab\|b)$	$=$	$G(aba\|ba)$	$=$	$aaba\|aba$
Aeolian	$\tilde{G}\tilde{G}\tilde{D}(a\|b)$	$=$	$\tilde{G}\tilde{G}(ab\|b)$	$=$	$\tilde{G}(aba\|ba)$	$=$	$abaa\|baa$

The following table lists the 6 plagal modes. In these (divided) words the letter b stands for whole step and the letter a stands for half step. It could be more plausible to exchange the two letters in each word by finally applying the morphism E with $E(a) =$

(6) These special Sturmian morphisms can be extended to automorphisms of the free group F_2 with two generators. The generated group is isomorphic to the Artin braid group B_4.

$b, E(b) = a$, which turns the listed special Sturmian morphisms into associated general ones, whose incidence matrices have determinant -1. This property algebraically reflects the musical markedness of the plagal modes with respect to the authentic ones. The Hypo-Locrian scale-pattern turns out to be amorphous.

Hypo-Ionian	$DDG(a\|b)$	$=$	$DD(a\|ab)$	$=$	$D(ba\|bab)$	$=$	$bba\|bbab$
Hypo-Dorian	$D\tilde{D}G(a\|b)$	$=$	$D\tilde{D}(a\|ab)$	$=$	$D(ab\|abb)$	$=$	$bab\|babb$
Hypo-Phrygian	$\tilde{D}\tilde{D}G(a\|b)$	$=$	$\tilde{D}\tilde{D}(a\|ab)$	$=$	$\tilde{D}(ab\|abb)$	$=$	$abb\|abbb$
Hypo-Lydian	$DD\tilde{G}(a\|b)$	$=$	$DD(a\|ba)$	$=$	$D(ba\|bba)$	$=$	$bba\|bbba$
Hypo-Mixolydian	$D\tilde{D}\tilde{G}(a\|b)$	$=$	$D\tilde{D}(a\|ba)$	$=$	$D(ab\|bab)$	$=$	$bab\|bbab$
Hypo-Aeolian	$\tilde{D}\tilde{D}\tilde{G}(a\|b)$	$=$	$\tilde{D}\tilde{D}(a\|ba)$	$=$	$\tilde{D}(ab\|bab)$	$=$	$abb\|babb$

The intermediate words in these generation processes of length 3 and 5 have a clear musical meaning as structural and pentatonic modes (see Subsection 1.2).

3.2. The Ionian Mode, Standardicity and Divider Incidence. — Those music theorists which are attracted by the fifth-generatedness of the diatonic scale as a possible source of explanatory power, are puzzled by the fact, that music history "chose" the Ionian mode for modern tonality rather than Lydian, where the finalis coincides with the origin of the generation. It is therefore interesting to inspect the Ionian mode from the view point of word theory. Figure 11 portraits the scale-step pattern and the scale folding of this mode.

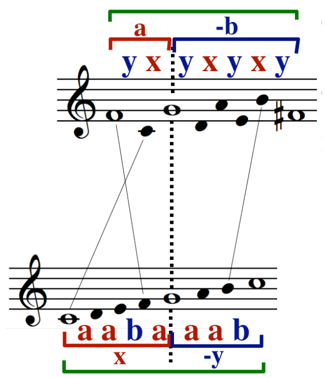

FIGURE 11. Portrait of the authentic Ionian mode

The following properties are shared by all authentic modes: The scale folding by fifth up and fourth down fills the width of an augmented prime. The augmented prime is divided in whole step up and half step down. The scale step-pattern by whole step up and half step up fills the height of an octave. The octave is divided in fifth up and fourth up.

The properties listed below are particularly noteworthy as they characterize the Ionian mode.

1. The divider of both words is the same tone G.
2. The leading tone B is the penultimate tone with respect to both words.
3. The central palindromes $u = aabaa$ and $v = yxyxy$ are both prefixes (and coincide with the Guidonian Hexachord C-D-E-F-G-A, which plays a prominent role in music theory).
4. The first tones C and F and the predecessor tones F and C of the common divider G are mutually exchanged.
5. The generating morphism is a *special standard morphism*, i.e. generated by the morphisms G and D.(7)

It can be shown that these properties are not just coincidental for the special case of the Ionian mode and the cardinality $n = 7$. They are all consequences of one another. The equivalence of divider incidence and special standardicity is proven in [**26**]. Figure 12 shows the strictly binary ramification of the special standard words. The transformational path towards the Ionian scale folding DGG reverses the transformational path towards the Ionian scale-step pattern GGD. These words coincide (up to renaming) with the directive words for the central palindromes. For plain adjoint for special standard words corresponds to an anti-automorphism of the monoid $\langle G, D\rangle$. The same fact is true for Christoffel words and the monoid $\left\langle G, \tilde{D}\right\rangle$, but not in general.(8)

4. Pitch Height and Beyond

The investigations of the Sections 2 and 3 are rather ignorant about specific musical intervals. What distinguishes a whole step $(2, 1)$ from a half step $(-5, 1)$ in the theory of well-formed modes is *not* a difference in pitch height: If we identify the second coordinates of these vectors with an abstract counting of generic height (such as the height on a staff in musical notation), then they are both steps of generic height 1. What distinguishes them, is the first coordinate, which we call generic *width.* It measures the amount of fifths or fourths in the folding. In the present section we draw a connection between the concepts of *pitch height*, *tone kinship* and the content of Sections 1 and 2.

(7) Be careful that the letters G and D homonymously denote diatonic tones and Sturmian morhisms.

(8) One may define a twisted analogue to the plain adjoint, which relates a to y and b to x. The twisted adjoint corresponds to Sturmian involution, which is an anti-automorphism that fixes G and $\tilde{G}$ and exchanges D and $\tilde{D}$. For twisted adjoints a similar result about divider incidence holds for words $w = f(xy)$, where $f \in \left\langle \tilde{G}, \tilde{D}\right\rangle \cdot G \cdot \left\langle \tilde{G}, D\right\rangle$ is a morphism which is composed of an anti-standard morphism, a copy of G and an anti-Christoffel morphism. See [**26**] for details.

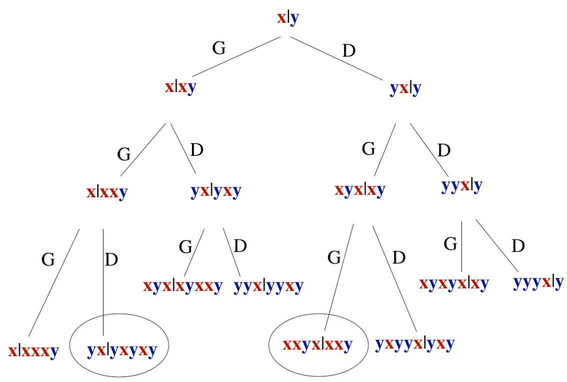

FIGURE 12. Binary Tree of special standard words according to the application of either D or G at each node.

4.1. Pitch Height and the Pythagorean Tone Lattice. — Many music theorists seem to believe that the ups and downs in pitch height are not sufficient to grasp the essence of musical tone relations. Music psychologists are able to provide empirical support to the assumption that tone relations in a tonal context involve fifth kinship as a constituent apart from the proximity relation in pitch height. However, recent articles related to mathematical music theory by Clifton Callender, Ian Quinn, Rahel Hall and Dmitri Tymoczko show that one may also shake rather interesting results out of a pitch-height-only approach (e.g. see [**37**]).

In the tradition of mathematical approaches to tone kinship since Leonhard Euler tone relations are described by means of finitely generated abelian groups ($\mathbb{Z}$-modules) whose generators represent musical intervals, such as octave, fifth, and major third which are considered to be constituents of elementary tone kinship. We restrict our discussion to the case of two free generators: fifth x and octave z. The $\mathbb{Z}$-module $P = \mathbb{Z}x \oplus \mathbb{Z}z$, – which is commonly known as the *Pythagorean tone lattice* – is of rank 2 over the ring of integers $\mathbb{Z}$. Each tone $ax + bz$ in this lattice has a unique pitch height $p(ax + bz) = log_2(3/2)a + b$. The linear form $p : P \to \mathbb{R}$ normalizes the pitch height ambitus of an octave to 1.[9]

In order to bring both concepts – pitch height and tone kinship – under one theoretical roof, it is crucial for the music-theoretical interpretation to distinguish between two mathematical concepts, namely of *rank of a free group* on the one hand and *dimension of a (real) vector space* on the other. The distinction leads to the following two alternatives:

[9] Guerino Mazzola (1990, 2002) made the proposal to study the linear pitch height as a linear form $p : \mathbb{Z}^3 \to \mathbb{R}$ on the Euler module, i.e. on a tone lattice that includes the third as a generating interval, aside of the octave and fifth. By embedding $\mathbb{Z}^3$ into $\mathbb{R}^3$ the linear form is associated with the pitch height vector $(1, log_2(\frac{3}{2}), log_2(\frac{5}{4}))$.

1. If one identifies the Pythagorean tone lattice P with its image $h(P) \subset \mathbb{R}$, one deals with a *dense* subgroup of rank 2 within the 1-dimensional real vector space $\mathbb{R}$.
2. If one identifies the Pythagorean tone lattice P with the subgroup of integral points (a, b) within the real plane one deals with a *discrete* subgroup of rank 2 within the 2-dimensional real vector space $\mathbb{R}^2$.

What could be music-theoretical reasons in favor of one or the other option? On the one hand one might feel uncomfortable with the fact that pitch height proximity in the dense subgroup does not properly reflect the kinship relations between tones. But on the other hand one should be puzzled by the redundancy of the 2-dimensional ambient space $\mathbb{R}^2$. What is the conceptual difference between the fifth x and the fifth-sized diminution of the octave $log_2(\frac{3}{2})z$? The difference $w = x - log_2(\frac{3}{2})z$ between these two vectors has pitch height zero and therefore spans the one-dimensional kernel of the linearly extended pitch height form to the real plane $\mathbb{R}^2$. The situation becomes even clearer if one considers the gradient $h = log_2(\frac{3}{2})x + z$ as one basis vector in $\mathbb{R}^2$ and w as another. The subspace $\mathbb{R}h$ can then be identified with the pitch height space and our discussion simply leads to the question whether the subspace $\mathbb{R}w$ has music-theoretical meaning or wether it is a candidate for Ockham's razor.

At this point Sturmian sequences come into play. The intersection of the constant zero pitch level $\mathbb{R}w$ with the Pythagorean lattice $\mathbb{Z}^2$ consists only of the origin $\{(0, 0)\}$. If view the subspace $\mathbb{R}w$ as an auxiliary line, we may approximate it in terms of a mechanical word - a discrete analogue for the zero pitch level. More precisely, we may consider the upper and lower mechanical words $\underline{s}, \bar{s} : \mathbb{N} \to \{0, -1\}$ with letters 0 and -1 (with negative slope $\alpha = -log_2(\frac{3}{2})$) (see the right part of Figure 13, which displays a section of these words, extended to both sides of the origin, i.e. as maps $\mathbb{Z} \to \{0, -1\}$).

$$\underline{s}(n) = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor \quad \text{and} \quad \bar{s}(n) = \lceil (n+1)\alpha \rceil - \lceil n\alpha \rceil.$$

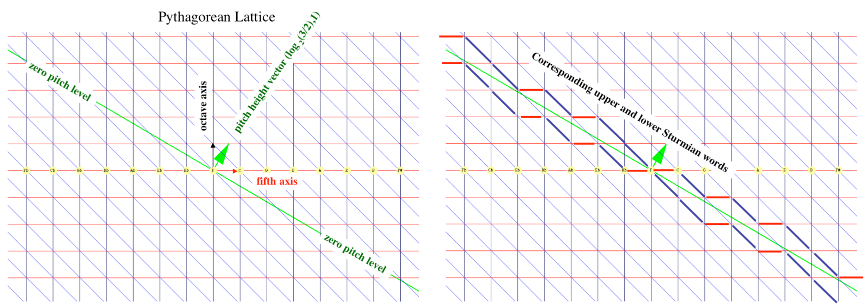

FIGURE 13. The Pythagorean Lattice

Figure 14 displays the upper mechanical word in a sheared manner, and gives a clearer picture of the situation. The left side of the upper mechanical word is equivalent to the right side of the lower mechanical word.

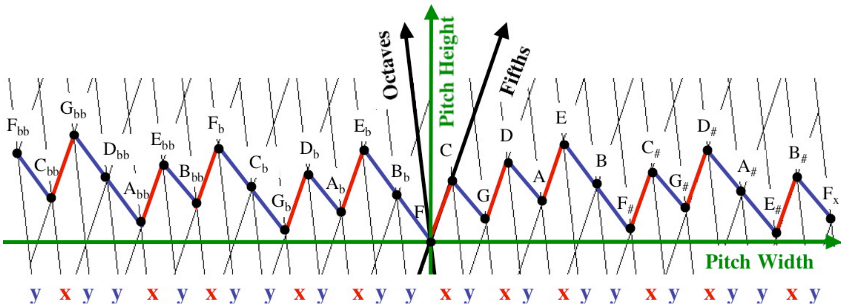

FIGURE 14. The infinite Pythagorean mechanical word. It renders the constant zero pitch level within the Pythagorean Tone Lattice. Its Christoffel prefixes correspond to well-formed tone systems.

Christoffel prefixes correspond to best approximations of the zero-pitch level. This provides the connection with the findings by Yves Hellegouarch, Norman Carey and David Clampitt, as discussed in subsection 1.2. Well-formed tone systems are local aspects of the infinite Pythagorean word. Modes come into play as soon as the zero pitch height level is shifted. Figure 15 displays the specific counterparts of the generic Lydian and Ionian modes, as studied in Sections 2 and 3.

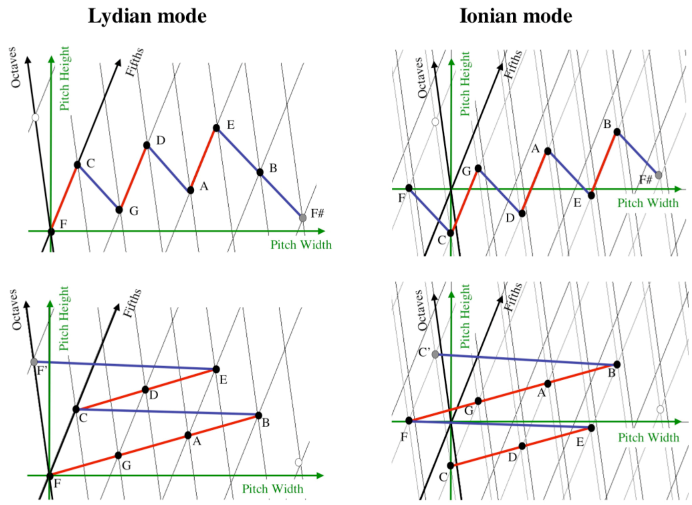

FIGURE 15. Well-formed factors of the infinite Pythagorean word are the specific manifestations of the associated generic modes.

4.2. Towards Music-theoretical Interpretation. — A thorough investigation of the possible music-theoretical meanings of the mathematical facts involves several directions of work. In [**15**] a philological approach is proposed, where selected unsolved problems in the connection between music theory and *tone psychology* are traced in the scholarly discourse. A particular anchor for this study is the Jacques Handschin's book [**27**] and its critical reception. With the concept of *tone character* Handschin addresses the individual tone qualities of the diatonic scale degrees. He explains them in terms of the relative positions along the width axis. A detailed discussion of Handschin's arguments and objections that have been raised by other authors shows that these arguments can be successfully sharpened with the help of the mathematical facts.

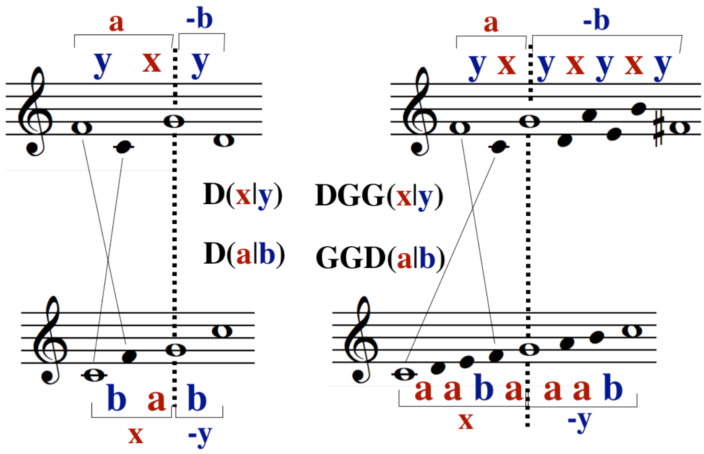

FIGURE 16. The three-tone scale on the left side corresponds to the *structural scale* in Figure 6. It has a minor-third comma. By comparison with the Ionian diatonic mode on the right side one may infer how the Christoffel-duality works in the three-note scale.

In [**24**] an analytical approach is proposed which interprets the progression of the fundamental bass in common practice harmonic tonality as a manifestation of fifth-fourth foldings. Particular attention is dedicated to a family of modes with three scale steps (see Figure 16). This approach mediates between concurring traditional theories of tonal harmony.

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