TILING THE (MUSICAL) LINE WITH POLYNOMIALS: SOME THEORETICAL AND IMPLEMENTATIONAL ASPECTS

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ABSTRACT

This paper aims at discussing the polynomial approach to the problem of tiling the (musical) time axis with translates of one tile. We show how this mathematical construction naturally leads to a new family of rhythmic tiling canons having the property of being generated by cyclotomic polynomials. We discuss the polynomial approach as an extension of the classical way of formalizing tiling canons by means of factorizations of cyclic groups and we show how both approaches are historically related to some mathematical conjectures (Keller, Minkowski, Fuglede). The implementation of the polynomial approach in the visual programming language *OpenMusic* offers to composers and music theorists a way of exploring huge catalogues of possible canons constructions. At the same time, the implementation shows that the representation of the tiling process with a canonical product of cyclotomic polynomials does not necessarily provide the musical solution that would naturally fit with the musical intuition.

1. INTRODUCTION

Tiling problems in music theory, analysis and composition have a relatively old history in mathematical music theory. From a pitch-perspective, the study of some tiling problems in music is historically related to Hugo Riemann original representation of the tone space by means of a translation of deformed squares generated by oblique major and minor thirds axis (see Figure 1).

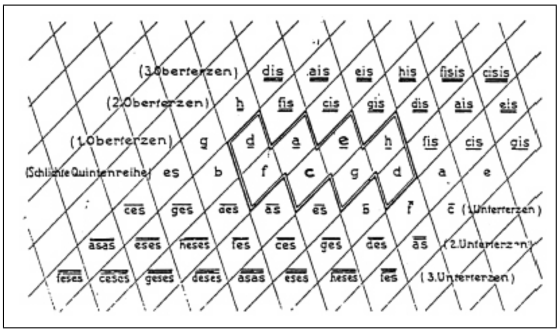

Figure 1. Hugo Riemanns *Tonnetz* (see [5]).

Many attempts have been made in order to provide some alternative models of the tiling process of the tone space, particularly by the so-called Neo-Riemannian American and European tradition. [1] The mathematical exploration of the tiling properties of the tone space naturally leads to computational models for computer-aided music analysis and composition, from C. Longuet-Higgins geometric representation of tonality in just-tuning intonation to the very recent *OpenMusic* implementation of harmonic transformations with the hexagonal tiling lattice model proposed by composer Jean-Marc Chouvel (See Figure 2).

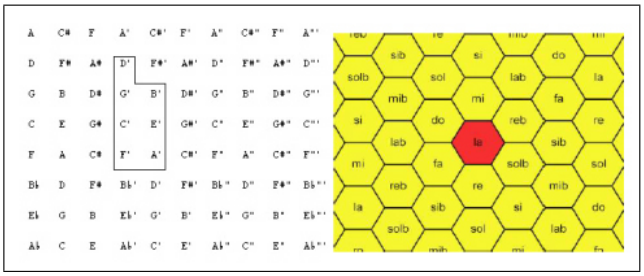

Figure 2. Longuet-Higgins (left) and Chouvel (right) tiling models of the tone space by translates of a square ([7]) and of a hexagon ([8]).

Surprisingly, despite the well-known canonical equivalence (isomorphism) between a well-tempered division of the octave and the cyclic character of any periodic rhythm [9], the study of some tiling properties of the time-line by means of translates of a given rhythmic tile (or some usual transformations of it) is a relatively new research area inside mathematical music theory. We have already sketched the history of the emergence of tiling rhythmic structures in music composition starting from Olivier Messiaen original attempt at defining musical canons independently of any considerations about pitch values [2]. Dan Tudor Vuza's algebraic model of tiling canon construction by the factorization of a cyclic group into a direct sum of two subsets [10] gave a strong impulse to the implementation of algebraic methods in music composition. [2] In this

[1] See [6] for a survey and a historical perspective on Neo-Riemannian theory from the standpoint of the American tradition.

[2] For a detailed presentation of the group factorization approach to the construction of tiling canons, together with the *OpenMusic* implementa-

paper we focus on a different approach to the tiling canon construction. This approach is based on the mathematical concept of cyclotomic polynomials and it enables to formalize a tiling rhythmic canon in terms of a factorization of the time-axis as product of cyclotomic polynomials of given order. Some preliminary definitions about cyclotomic polynomials, tiling of the line process and rhythmic tiling canon construction will be provided in Section 2. In Section 3 we show how this approach has been implemented in *OpenMusic* visual programming language and discuss some difficulties in directly applying the cyclotomic factorization to the canon construction. In Section 4 we discuss some interesting connections between Vuza's original model of *Regular Complementary Canons of Maximal Category* [10] and the polynomial approach by also showing how both strategies are intimately related to some mathematical conjectures.

2. SOME PRELIMINARY DEFINITIONS

This section provides some definitions on cyclotomic polynomials and some general factorization theorems.

2.1. 0-1 polynomials and rhythmic canons

A rhythmic canon is a decomposition of a cyclic group Z_n into a direct sum of two subsets:

$$Z_n = A \oplus B$$

An enhancement of the ambient structure originates to [11]. By putting $A(x) = \sum_{a\in A} x^a$, the above equation becomes a relation between 0-1 polynomials, that is to say polynomials with coefficients being either 0 or 1:

$$A(x) \times B(x) \equiv 1 + x + x^2 + \cdots + x^{n-1} \pmod{x^n - 1}$$

Factors of the polynomial $\Delta_n(x) = 1+x+x^2+\cdots+x^{n-1}$ are thus of paramount importance, especially those with 0-1 coefficients. We find a number of them by considering the notion of cyclotomic polynomials.

2.2. Cyclotomic polynomials

Definition 1 *The n^{th} cyclotomic polynomial is*

$$\Phi_n(x) = \prod_{\gcd(k,n)=1} (x - e^{2i\,k\,\pi/n})$$

This is the monic polynomial whose roots are the primitive units of order n, that is to say the $\xi \in C$ for which $z^n = 1$ though $z^r \neq 1$ for $1 \leq r < n$.

A classical result states that these polynomials have integer coefficients, i.e. $\Phi_n(x) \in Z[x]$.

Another classical result states that they are irreducible in the euclidean ring $Q[x]$, and hence in $Z[x]$. In other words, any polynomial in $Z[x]$ having a primitive unit root of order n has Φ_n as a factor.

tion, see [18]. For a combinatorial discussion of Vuza's model, also see [20]

Directly relevant to rhythmic canons is the fact that the product of a selection of cyclotomic polynomials can be expressed by the following equations:

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$
$$\Delta_n(x) := 1 + x + \cdots + x^{n-1} = \prod_{d|n, d\neq 1} \Phi_d(x) \quad (1)$$

Formula (1) enables to compute efficiently all cyclotomic polynomials.

Usually Φ_n have small coefficients, often $0, 1$ or -1. In particular the repetitive rhythm of a metronome is easily expressed as a product of cyclotomic polynomials:

$$1+x^k+x^{2k}+\ldots x^{(m-1)k} = \frac{x^{mk}-1}{x^k-1} = \prod_{d|mk, d\nmid k} \Phi_d(x)$$

Note that some of these cyclotomic factors are NOT 0-1 polynomials, though their product is. For instance:

$$\Phi_6 \times \Phi_3 = (1-x+x^2)(1+x+x^2) = 1+x^2+x^4$$

2.3. CYCLOTOMIC POLYNOMIALS AND THE RHYTHMIC TILING CANON CONSTRUCTION

The importance of these particular polynomials lies in the following lemma:

Lemma 1 *If $A \oplus B = Z_n$, then for all $d \mid n$ ($d \neq 1$) Φ_d is a divisor of either $A(x)$ or $B(x)$.*

Thus, cyclotomic polynomials occur in all rhythmic canons. Conversely, in the *OpenMusic* implementation we have tried to use these polynomials to build up some rhythmic canons. As we will see, some other rhythmic canons are left out of this schema, but nevertheless it gives an interesting degree of control over the canons construction.

Except in special cases [2], it is not known whether a given rhythmic motif enables to make a canon unless one is able to exhibit such a canon; but looking for the outer rhythm knowing only the inner rhythm is impractical as far as computing time is concerned.

Since 1998, there is a useful sufficient condition (also necessary when the number of notes in the motif has at most two prime factors) dealing with the cyclotomic factors. Let A be an inner rhythm and $A(X)$ the associated 0-1 polynomial. Put $R_A = \{d, \Phi_d \text{ divides } A(x)\}$ and $S_A = \{p^\alpha \in R_A\}$ the subset of prime powers. It is proved in [15] that if both following conditions are true, then A enables to build a rhythmic canon:

$(T_1) : A(1) = \prod_{p^\alpha \in S_A} p$

$(T_2) : p^\alpha, q^\beta \cdots \in S_A \Rightarrow (p^\alpha, q^\beta \ldots) \in R_A$

Condition (T_1) is also always necessary. We will look again at these conditions in connection with the spectral conjecture (or Fuglede conjecture) and its relationship with Vuza canons in the final section. The next section deals with construction of rhythmic canons in *OpenMusic* using these conditions.

3. THE OPENMUSIC IMPLEMENTATION

3.1. Simple cases of tiling canon constructions via cyclotomic factors.

In the simpler cases, e.g. for small periods, we do not need to test if a given product of cyclotomic polynomial verifies (or not) the two conditions by Coven-Meyerowitz. We simply use equation (1) to express the inner rhythm as a product of cyclotomic factors, and the outer rhythm as the product of the remaining ones. Figure 3 shows the list of solutions for all periods $4 \leq n \leq 10$. (We did not include the cases of periods equal to a prime number p. In this case, the tiling canon trivially reduces to a single subset $(1\ 1\ \ldots 1)$ of the time-axis having cardinality equal to p).

```
;============PERIOD 4===========
'(4
(((1 1 1 1) nil (2 4)))
(((1 1) (1 0 1) (2)))
(((1 0 1) (1 1) (4)))
)
;============PERIOD 6===========
'(6
(((1 1) (1 0 1 0 1) (2)))
(((1 1 1) (1 0 0 1) (3)))
(((1 0 0 1) (1 1 1) (2 6)))
(((1 0 1 0 1) (1 1) (3 6)))
)
;============PERIOD 8===========
'(8
(((1 1 1 1 1 1 1 1) nil (2 4 8)))
(((1 1) (1 0 1 0 1 0 1) (2)))
(((1 0 1) (1 1 0 0 1 1) (4)))
(((1 0 0 0 1) (1 1 1 1) (8)))
(((1 1 1 1) (1 0 0 0 1) (2 4)))
(((1 1 0 0 1 1) (1 0 1) (2 8)))
(((1 0 1 0 1 0 1) (1 1) (4 8)))
)
;============PERIOD 9===========
'(9
(((1 1 1 1 1 1 1 1 1) nil (3 9)))
(((1 1 1) (1 0 0 1 0 0 1) (3)))
(((1 0 0 1 0 0 1) (1 1 1) (9)))
)
;============PERIOD 10===========
'(10
(((1 1) (1 0 1 0 1 0 1 0 1) (2)))
(((1 1 1 1 1) (1 0 0 0 0 1) (5)))
(((1 0 0 0 0 1) (1 1 1 1 1) (2 10)))
(((1 0 1 0 1 0 1 0 1) (1 1) (5 10)))
)
```

Figure 3. Catalogue of tiling rhythmic canon that are directly given by simple cyclotomic factors (or any given product of them).

We briefly analyze the case of period $n = 8$ which is the only one to possess three cyclotomic polynomials Φ_2, Φ_4 and Φ_8 that can be combined in all possible $3!$ ways.

3.1.1. The case n=8

This case shows the strong symmetric character of the tiling construction principle for a period n having three divisors. Each cyclotomic polynomial associated to a given divisor (e.g. Φ_4) tiles the line, i.e. it can be taken as a inner rhythm for a tiling rhythmic canon. The outer rhythm is automatically given by the product of remaining cyclotomic polynomial (in this case $\Phi_2 \times \Phi_8$). In order to build the corresponding tiling canon we simply need the lisp-function *poly2canon* that canonically associates a given cyclotomic polynomial to a subset of the cyclic group Z_n. The following figure (Figure 4) shows the tiling process starting from cyclotomic factors. On the left we use the circular representation to represent the tiling canon as a direct sum of the two subsets associated respectively with cyclotomic polynomials Φ_4 and $\Phi_2 \times \Phi_8$. Notice that a direct sum of two subsets can also be expressed in terms of transpositional combination [6]. This is why we use the lisp function *transp-comb*. On the right we use the standard canons function of the OM-Libray Zn to explicitly build the 4 voices canon (cf. [2]).

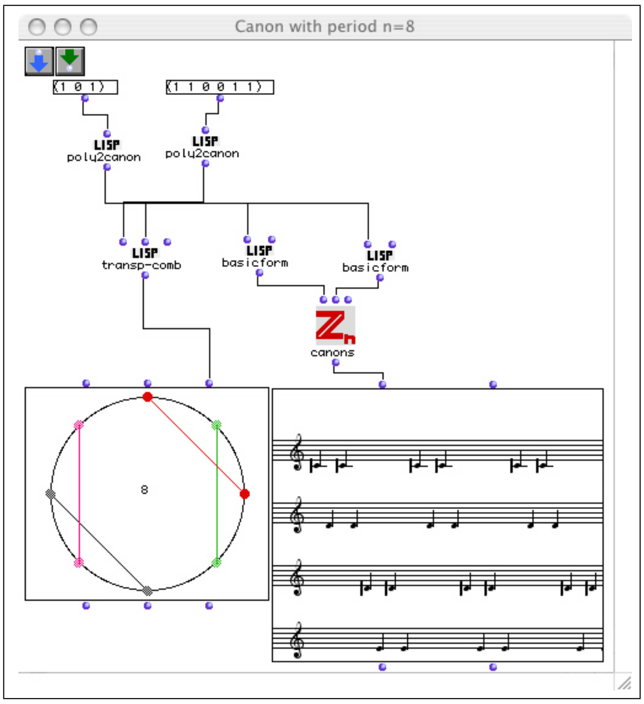

Figure 4. A tiling rhythmic canon obtained by cyclotomic factors.

Thanks to the symmetric role of the cyclotomic factors we can easily apply the same cyclotomic decomposition to construct the dual rhythmic canon, i.e. the 2 voices canon, each voice being associated to the subset of Z_8 having 4 elements. The dual case is represented in Figure 5. Although one can always construct a new canon by dualizing an existing factorization of a given cyclic group into two subsets, this process no longer applies to the cyclotomic factors, as we show in the next section.

3.2. The list of cyclotomic products having the T1 and T2 properties

For periods $n > 10$ we need to test the two conditions of Coven and Meyerowitz in order to be sure that the associated rhythmic canon tiles the time axis. The algorithm thus tests all sublists of the list of divisors of n for conditions T_1, T_2 and computes the corresponding products of cyclotomic polynomials.

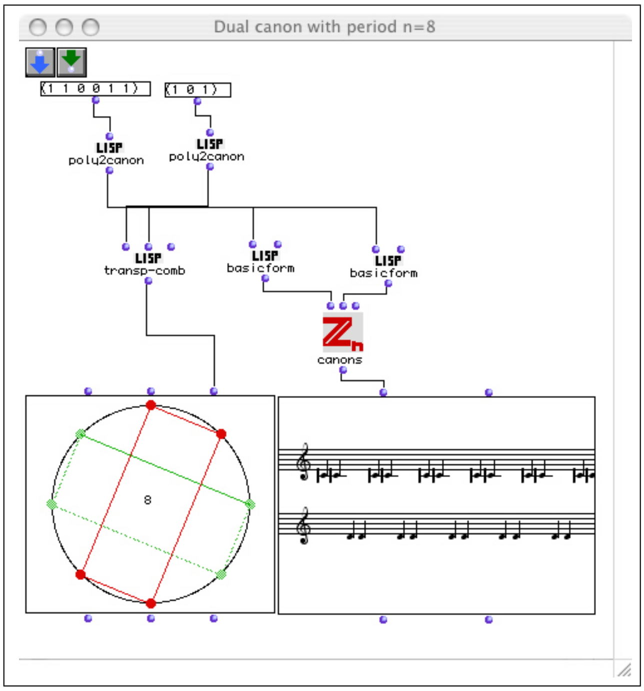

Figure 5. The dual tiling rhythmic canon obtained by inverting the role of cyclotomic factors for $n = 8$.

We will now discuss in detail the cases $n = 12$ and $n = 16$. The first case shows some difficulties in obtaining symmetrically inner and outer rhythms as products of cyclotomic factors. The second case enables to understand some general properties of the "modulation" process between rhythmic tiling canons having period n and $2n$.

3.2.1. *The case n=12*

Since 12 has five divisors $(2, 3, 4, 6, 12)$, the cyclotomic factors may combine in several ways in order to tile the line. Apart from $\Phi_6 = 1 - x + x^2$ and $\Phi_{12} = 1 - x^2 + x^4$ (which are not 0-1 polynomials), each cyclotomic polynomial of index $k \in \{2, 3, 4\}$ tiles the line by itself.

We lose somehow the symmetric distribution of the simpler cases. For instance $A(x) = \Phi_2\Phi_4\Phi_6$ should tile, as conditions T_1, T_2 are fulfilled: $S_A = \{2, 4\}$ and there is only (T_1) to check, namely $A(1) = 2 \times 2$.

But the simple trick of multiplying the remaining cyclotomic factors of $1 + x + x^2 + \ldots x^{11}$ does not work this time, as

$$\Phi_3\Phi_{12} = 1 + x - x^3 + x^5 + x^6$$

is not a 0-1 polynomial.

The outer rhythm may still be produced with cyclotomic polynomials, following the proof of the Coven and Meyerowitz theorem, but the formula is more complicated:

$$B(x) = \Phi_3(x^4) = 1 + x^4 + x^8 = \Phi_3\Phi_6\Phi_{12}$$

(see figure 6)

As usual, we can build the dual canon by simply inverting the role of the factors (see Figure 7).

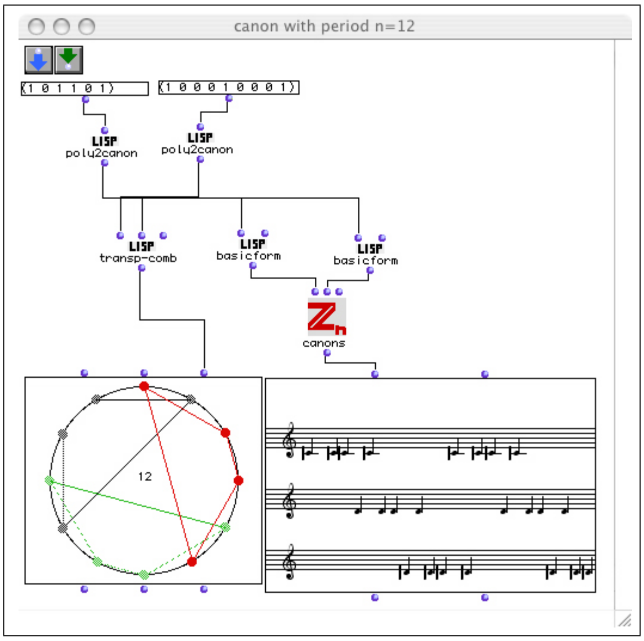

Figure 6. A 3 voices tiling rhythmic canon associated with the two polynomials $A(x)$ and $B(x)$.

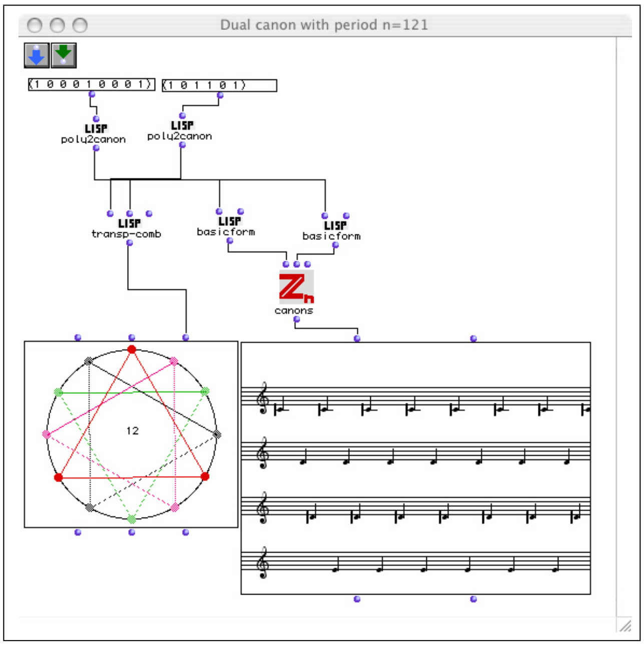

Figure 7. A 4 voices tiling rhythmic canon constructed by taking the polynomial $B(x)$ as inner rhythm and $A(x)$ as outer rhythm.

3.2.2. *The case n=16*

This case is interesting in relation to the case $n = 8$ that we have studied in section 3.1.1. Intuitively, when we multiply a given period n by some integer factor k, all inner rhythms that tile Z_n will necessarily tile Z_m with $m = nk$ (although with respect to a different outer rhythm). All possible solutions for period $n = 16$ are given in Figure 8.

As we can see, there is the same symmetry principle in

```
;=============PERIOD 16============
'(16
(((1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1) nil (2 4 8 16)))
(((1 1) (1 0 1 0 1 0 1 0 1 0 1 0 1 0 1) (2)))
(((1 0 1) (1 1 0 0 1 1 0 0 1 1 0 0 1 1) (4)))
(((1 0 0 0 1) (1 1 1 1 0 0 0 0 1 1 1 1) (8)))
(((1 0 0 0 0 0 0 0 1) (1 1 1 1 1 1 1 1) (16)))
(((1 1 1 1) (1 0 0 0 1 0 0 0 1 0 0 0 1) (2 4)))
(((1 1 0 0 1 1) (1 0 1 0 0 0 0 0 1 0 1) (2 8)))
(((1 1 0 0 0 0 0 0 1 1) (1 0 1 0 1 0 1) (2 16)))
(((1 0 1 0 1 0 1) (1 1 0 0 0 0 0 0 1 1) (4 8)))
(((1 0 1 0 0 0 0 0 1 0 1) (1 1 0 0 1 1) (4 16)))
(((1 0 0 0 1 0 0 0 1 0 0 0 1) (1 1 1 1) (8 16)))
(((1 1 1 1 1 1 1 1) (1 0 0 0 0 0 0 0 1) (2 4 8)))
(((1 1 1 1 0 0 0 0 1 1 1 1) (1 0 0 0 1) (2 4 16)))
(((1 1 0 0 1 1 0 0 1 1 0 0 1 1) (1 0 1) (2 8 16)))
(((1 0 1 0 1 0 1 0 1 0 1 0 1 0 1) (1 1) (4 8 16)))
)
```

Figure 8. Catalogue of tiling rhythmic canons of period $n = 16$ that are directly given by simple cyclotomic factors (or any given product of them).

the distribution of solutions for the inner and outer rhythm. Take for example the solution given by the cyclotomic polynomial Φ_4. In the previous case, with $n = 8$, the outer rhythm was provided by the product $\Phi_2 \times \Phi_8$. By multiplying the period by 2, these two factors will still be present in the new outer rhythm that will be enlarged by the remaining cyclotomic factor, i.e. Φ_{16}. The figure 9 shows the new canon having inner and outer rhythms given by Φ_4 and $\Phi_2 \times \Phi_8 \times \Phi_{16}$ respectively.

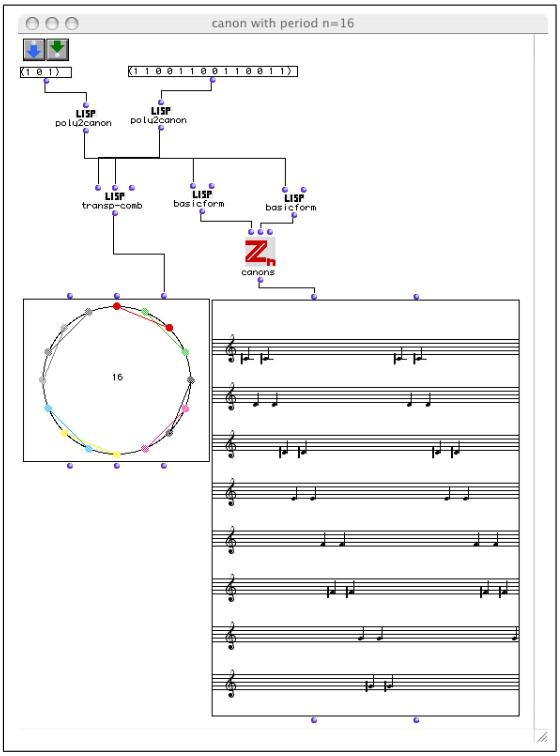

Figure 9. A 8 voices tiling rhythmic canon associated with the two polynomial Φ_4 and $\Phi_2 \times \Phi_8 \times \Phi_{16}$

Notice that we can interpret the relation between the previous case ($n = 8$) and the new case ($n = 16$) as a rhythmic "modulation" process enabling to transform a 4 voices canon into a 8 voices canon being generated by the same rhythmic pattern. This technique has been largely applied by composer Georges Bloch in the case of a special family of rhythmic tiling whose inner and outer rhythm do not have transpositional symmetry (i.e. they are not interpretable, from a pitch perspective, as examples of Messiaen limited transposition modes).[3] It is a good example of purely mathematical property (namely, if p divides d then $\Phi_d(x^p) = \Phi_{pd}(x)$) which takes a new meaning when used by a musician. We now go back to this special case in order to show have both approaches (group factorization and cyclotomic decomposition) are intimately linked to some mathematical conjectures.

4. RHYTHMIC TILING CANONS AND SOME MATHEMATICAL CONJECTURES

As we said, Vuza's original model of rhythmic canons focused on a very special family of tiling canons. These were group-theoretically formalized as the solution of factorizing a given cyclic group into the direct sum of two non-periodic subsets. This theory has been established by Vuza independently of any consideration of geometric tiling conjectures. Nevertheless, as we have already shown [18], it is possible to directly link Vuza's model to Minkowski's original conjecture of the tiling of the n-dimensional space by unit cubes. We briefly summarize this connection in order to show how Minkowski's conjecture could provide a bridge between Vuza's model and the new tiling constructions by cyclotomic polynomials.

4.1. Minkowski's Conjecture and Vuza's model

In the introduction, we have mentioned some examples of tiling of the pitch-tone space by polygons (rhombus, squares, hexagons). If we focus on squares, we easily see that in any planar tiling by squares, at least a couple of squares have an entire edge in common. According to Stein and Szabo [21], we call this property the "twin property" (See Figure 10).

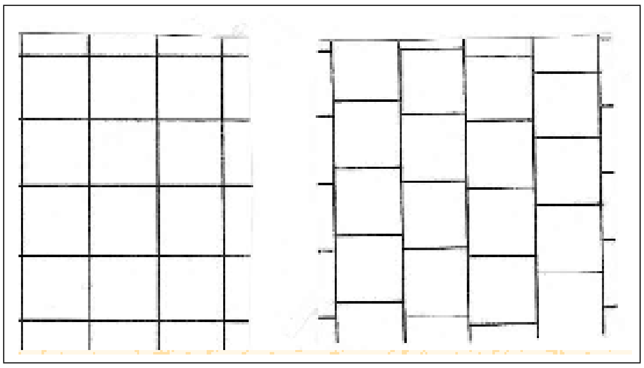

Figure 10. Two examples of planar tiling having the twin property.

[3] The modulation process between *regular complementary canons of maximal category* has been applied in the composition for small ensemble *Projet Beyeler* (2001) which were commissioned by the Beyeler Foundation in Basel. For an analytical account of Georges Bloch's compositional approach, see [17].

Hermann Minkowski conjectured that the "twin property" would be true in any dimension. More rigorously:

Minkowski's Conjecture (1907). Any lattice tiling of n-dimensional space by unit cubes has the twin property, i.e. there must be a pair of cubes (the twins) that share a complete (n-1)-dimensional face [22].

By translating (and finally solving) the geometric conjecture into a group-theoretical problem, mathematician G. Hajos opened the path to the classification of so-called Hajos groups, i.e. groups such that for any factorization into finite subsets, at least one of the subset is periodic. [4] One can easily see that Vuza's original condition of maximal category does not apply to Hajos groups. Moreover, one could try to develop a theory of Hajos (and non-Hajos) groups by starting from a weaker version of Minkowski's conjecture, for example by removing the lattice condition. This generalization has been suggested by Keller by stating that any tiling of the n-dimensional space by parallel unit cubes must have the "twin property" [16]. The conjecture is still open for dimensions $6 < n < 10$, as well as the musical relevance of the generalized model.

4.2. Vuza canons and the spectral conjecture

The operation of concatenation relating period 8 to period 16 canon examplifies the specificity of Vuza canons: a Vuza canon is precisely a canon that is not obtainable by concatenation of copies of a smaller canon. Enumeration techniques inspired by Polya and Burnside combinatorial algebra enabled mathematician H. Fripertinger to list all Vuza canons for periods 72 and 108; it is thus now known that Vuza canons are scarcer than one out of a million. Nevertheless, they are a key to the famous spectral conjecture [12] still unsolved in dimension 1. This conjecture, not unlike the conjectures by Minkowski, Hajos and Keller, links geometry to harmonic analysis. It was noticed shortly after the Coven-Meyerowitz discovery that conditions $(T_1), (T_2)$ are strongly linked to the spectral condition for a tiling: if a motif is spectral then (T_1) is true, if furthermore (T_2) is true then the motif is spectral [13].

But a canon that is NOT Vuza may be decomposed into a smaller canon, by the inverse operation of concatenation. For instance, motif $(0, 1, 4, 5)$ tiling with period 8 is two copies of motif $(0, 1)$ which tiles Z_4.

It was proved [3] that this operation preserves condition (T_2). Hence if a rhythmic canon tiles without being spectral, it shall be reduced to a Vuza canon with the same property. In other words:

If the spectral conjecture is false then it is false for some Vuza canon.

If all Vuza canons are spectral then so are all canons of any kind.

The scarcity of Vuza canons is a further indication that the spectral conjecture might be true in dimension 1. Furthermore, all the algorithms currently used for producing Vuza canons are guaranteed to produce canons verifying (T_2) (from other kinds of reduction techniques). But there is as yet no certainty as to the general Vuza canon, for there is no known way to produce or descript the cyclotomic structure of all of them.

[4] Musically speaking, at least one of the factors has Messiaen's transposition limited property.

5. CONCLUSIONS

The *Open Music* implementation of the cyclotomic approach is a necessary step in our general search of all possible solutions of tiling rhythmic canons of a given period. We have shown some connections between the classical approach based on the factorization of a cyclic group into two subsets and this new approach that makes use of the mathematical theory of tiling the line by translated of a cyclotomic polynomial (or of a product of cyclotomic polynomials). Coven-Meyerowitz conditions have been the starting point for implementing compositional algorhitms enabling to tile the musical line via cyclotomic polynomials. Although the two conditions are also necessary in some special cases, the problem remains of establishing necessary and sufficient conditions for the tiling of the line problem. We have shown some difficulties in trying to recover Vuza's original theory by the cyclotomic decomposition (and vice-versa). As in the case of the classical approach of rhythmic tiling canons construction by factorization of cyclic groups, the polynomial approach naturally leads to some still open mathematical conjectures, as the spectral conjecture. We suggest that the special family of *Regular Complementary Canons of Maximal Category*, originally conceived by Vuza in terms of factorizations of cyclic groups into non-periodic subsets, could play a major role in the musical interpretation (and mathematical solution) of the spectral conjecture.

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