A Unified Theory of Chord Quality in Equal Temperaments

by

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To the memory of David Lewin

CURRICULUM VITÆ

Ian Quinn was born at Warner-Robins AFB in Georgia on 19 March 1972. He was awarded the B.A. in music from Columbia University in 1993. After studies in the Ph.D. Program in Music at the CUNY Graduate Center under a Robert E. Gilleece Fellowship, he came to Eastman in 1996 with a Robert and Mary Sproull Fellowship from the University of Rochester, and was named Chief Master's Marshal in 1999, having earned the M.A. in October 1998. In that same year he won the Edward Peck Curtis Award for Excellence in Teaching by a Graduate Student. He has held teaching assistantships at Eastman and instructorships both at Eastman and at the University of Rochester's College Music Department. He left residence at the University to take the first postdoctoral fellowship in music theory at the University of Chicago (2002–03) and, subsequently, faculty appointments at the University of Oregon (2003–04) and Yale University (from 2004). He has performed on Nonesuch and Cantaloupe Records with Ossia, an ensemble he cofounded in 1997 that continues to play an active role in the new-music scene at Eastman. His research as a graduate student has been presented at many conferences and published in *Music Theory Spectrum*, *Music Perception*, and *Perspectives of New Music*.

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right way once I'd fleshed out that idea, I waited too long to ask. So it's with deep sadness, and more than the usual humility, that I add the boilerplate disclaimer: any remaining errors are entirely my responsibility.

ABSTRACT

Chord quality — defined as that property held in common between the members of a pcset-class, and with respect to which pcset-classes are deemed similar by similarity relations (interpreted extensionally in the sense of Quinn 2001) — has been dealt with in the pcset-theoretic literature only on an ad hoc basis. A formal approach that generalizes and fuzzifies Clough and Douthett's theory of maximally even pcsets successfully models a wide range of other theorists' intuitions about chord quality, at least insofar as their own formal models can be read as implicit statements of their intuitions. The resulting unified model, which can be interpreted alternately as (a) a fuzzy taxonomy of chords into qualitative genera, or (b) a spatial model called Q-space, has its roots in Lewin's (1959, 2001) work on the interval function, and as such has strong implications for a unification of general theories of harmony and voice leading.

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INTRODUCTION

The commonplace notion that a sonority can be "closely" or "distantly" related to other sonorities evokes a metaphorical space, in which individual sonorities are distributed according to what we will **chord quality**. Closely and distantly related sonorities are literally close to and distant from one another, respectively, in such a space; a sonority's quality can be defined as its location in that space. From a theoretical point of view, an understanding of the nature of chord quality might take the form of a description of that space's structure and the laws that determine a sonority's position. This is our main goal; we will refer to the space as **Q-space.**

I have shown elsewhere (Quinn, 2001) that the various numerical models generally characterized as pcset-class similarity relations agree with each other to a high degree, despite major differences in their internal workings. In the present context, it is helpful to think of the numbers put out by similarity relations as estimates of distances between sonorities in Q-space. This approach accounts both for the agreement among various similarity relations (since they are, in principle, estimating distances in a single space) and for their failure to converge completely with one another (since each is only an estimate).

Likewise, it is not difficult to conceive of the pcset-theoretical tools that produce taxonomic categories superordinate to the pcset-class as modeling structural features of Q-space. We might expect, for instance, that the pcsets belonging to a single one of Forte's genera (1988) would lie near one another in Q-space, and that the system of genera as a whole can be viewed as a system of overlapping regions in Q-space. The same could be said of Forte's pcset-complexes (1973), of any of Morris's set-group systems (1982), or of Hanson's categories (1960). Again, the spatial metaphor can be of assistance in understanding the nature of the differences among all of these systems. For an example of the different ways in which it is possible to describe different regions in the same underlying space, one need only consider the significant ways in which the map of Europe changed over the course of the twentieth century — centuries-old villages in (say) Macedonia or Transylvania have been incorporated into many different states and spheres of influence without significantly changing certain "natural" and "transcendent" national properties of those villages. Such properties tend to be shaped primarily by permanent geographical features such as bodies of water, deserts, or mountain ranges.

In Chapter 1 we will survey the approaches of Hanson, Forte, Morris, and many others, and see that certain well-known sonorities (e.g., the diatonic and pentatonic collections, Messiaen's modes of limited transposition, the hexatonic scale) turn up as what we will call **prototypes** — highly characteristic sonorities, relative to which the quality of other sonorities is determined. As such, these prototypical sonorities can be thought of as sitting atop the mountains of Q-space, of which we will see that there are six, roughly corresponding to the six interval classes of twelve-tone equal temperament. Scattered around the slopes of these mountains are other sonorities more or less close to the prototypes atop nearby peaks. What we are referring to here as a mountain will be called, more technically, a **qualitative genus**, defined as a taxonomic entity characterized by prototypical sonorities and encompassing sonorities to variying degrees, according to their closeness to the prototypes. The closeness or distance of an arbitrary sonority to the prototypes of a qualitative genus will be described in terms of the **intrageneric affinities** of the genus, and abstract structural relationships among genera will be called **intergeneric affinities**.

Chapter 2 will be concerned with generalizing the framework just described, in which qualitative genera are characterized in terms of prototypes and affinities. Clough and Douthett's theory of maximally even sets (1991) lies at the heart of this generaliztion, since (to oversimplify only slightly) each maximally even set, together with its complement, is a prototype of a unique qualitative genus. We will see that the association of qualitative genera with ME sets is more productive than associating them with interval classes because of certain antinomies arising in equal-tempered pitch-class universes other than the usual one with twelve pcs. Throughout this work we will give qualitative genera names of the form $\mathbb{Q}(c, d)$, where c is the number of pcs in the universe, and d is the cardinality of the ME set prototypical of the genus. Although this nomenclature will be slightly uncomfortable in Chapter 1, its utility will be amply demonstrated in Chapter 2.

In Chapter 3 we will generalize the framework once again, making a connection to Lewin's work on the interval function (1959; 2001). This will allow us to comprehend more easily the structure of Q-space and its utility as a locus for analytical discourse. More importantly, the depth of the mathematical connections shown to exist between Lewin's work and that of Clough and Douthett, as well as to the wide range of pcset-theoretic work discussed in Chapter 1, suggests that the theory of Q-space is a powerful starting point for future theoretical development — particularly in the direction of understanding the relationships among the abstract theories of harmony and voice leading that constitute the landmarks of recent pcset-theoretic research.

CHAPTER 1

Theoretical background

Suppose we wish to make a "purely" harmonic analysis of a piece of music. Regardless of the musical idiom, this sort of analysis essentially involves two steps: first, the identification of chords in the musical surface; and second, the assertion of relationships among those chords.

Usually, such an analysis results in a taxonomic interpretation of the chords involved, organizing them conceptually into categories akin to those Forte (1988) calls *species* and *genera*. A tonal piece might contain, for instance, an A minor seventh chord that is analyzed as an exemplar of the species of II^7 chords and the subdominant genus. An atonal piece analyzed in Forte's early pcset-complex system, on the other hand, might contain the pcset {03469}, an exemplar of the species 5–31 [01369], whose generic affiliation is the Kh-complex about the octatonic collection (or its complement, a proviso we will assume in all subsequent mentions of a pcset-complex). One major difference between the tonal and atonal cases, of course, is that while the functional genera of tonal theory have a syntactic character, there is no good evidence to support a general syntactic theory of atonal harmony. But leaving the question of syntax aside, it is notable that many theorists have concerned themselves with the question of how atonal harmonic genera are constituted; in addition to the specific system of categories that Forte calls "genera," we also have Hanson's (1960) "great categories," Forte's (1973) "pcset-complexes," Morris's (1982) "set-groups," and Eriksson's (1986) "regions." Two other authors have developed ad-hoc taxonomic systems in connection with the music of particular composers: Parks (1989) proposes a family of what he also calls "genera" for the music of Debussy; and Headlam (1996) works out, in his third chapter, a related system of chord classification for the atonal music of Berg.

All of these theories engage what we might think of as a fivefold hierarchy of increasingly general conceptual entities engendered by a piece of typical Western art music that is being conceived harmonically: (0) the sounding music, (1) the notated music; (2) the pcset or chord; (3) the species; and (4) the genus. Each level of this hierarchy abstracts essential harmonic features away from more accidental features of the previous level. For example, the analytical utility of the score (level 1) depends on the conceit that there is something essential we wish to analyze that is unaffected by accidents of acceptable performance (level 0); one might refer to that thing as "the music itself." Similarly, the analytical utility of the pcset (level 2) depends on the conceit that there is something we wish to analyze that is unaffected by the registration, timbre, and temporal order of the musical formation (level 1); one might refer to that thing as "harmony." This kind of framework, which highlights the common music-theoretic assumption that there are levels of essential and accidental features of pitch structure, can be helpful in thinking about how the higher levels figure into the picture.

A nominalist might say that a pcset-class is nothing more than an equivalence class of pcsets under transposition and inversion, without justifying the assertion of the relationship among pcsets and pcset-classes in terms external to the theory. Indeed, in *The Structure of Atonal Music* and most of his other writings, Forte meticulously avoids turns of phrase that might signal any deviation from such a nominalist standpoint, preferring to let the facts of the theory — that one may find set-complex coherence among the pcsets and pcset-classes of an atonal work — speak for themselves. In a sense, the outcome seems to be sufficient to justify, for Forte and for those who promulgate related theories, the assumption of equivalence under transposition and inversion, an assumption with a long and distinguished history that has been traced elsewhere (Bernard, 1997; Nolan, 2002).

This observation should not be taken as a critique of Forte's nominalist position. Yet one cannot avoid the suspicion that pcset-classes would not have been quite so historically resilient had they been founded on some other form of equivalence. Suppose two pcsets were to be considered equivalent if their constituent pcs, under integer notation, sum to the same quantity modulo 12. A theory founded on such a definition of equivalence, regardless of the kinds of coherence one can find with it, is not likely to achieve currency even among music theorists. Writers of textbooks, who are addressing the somewhat tougher audience of undergraduates, are more likely to use turns of phrase such as those Forte eschews, and thus we find, *inter alia*, Joseph Straus claiming in his widely used textbook that "the mere presence of many members of a single set class guarantees a certain kind of *sonic unity*" (2000, p. 49, emphasis added). We may take this as meaning, essentially, that there is something we wish to analyze that is unaffected by transposition and inversion of pcsets (level 2), not to mention the registration, timbre, and temporal order of the musical formation (level 1), or the subtleties of acceptable performance (level 0), and that its potential "sonic" significance (whatever we take that to mean) of this thing, which we might refer to as **chord quality**, is what motivates the concept of the pcset-class.

Chord quality, then, can be defined nominally — provided at least that one believes in properties — as that property that is held in common between all members of any pcset-class, and that property by which various pcset-classes are distinguished from one another to varying degrees. It takes its place in the hierarchy of variously essential and accidental properties that is structured by what philosophers call *supervenience*: Property A supervenes on Property B if and only if any change in Property A necessarily entails a change in Property B. To assert that properties of chord quality supervene on properties of harmony, which supervene on properties of "the music itself," is to say that one can change a harmony (by transposing or inverting it) without changing its quality, but one cannot change a harmony without changing "the music itself." It can be helpful to think of a supervenient property as an abstraction of certain aspects or facets of those properties on which it supervenes.

Within this framework, we can conceive of the aforementioned theories of chord genera (broadly construing the term) as defining various properties that supervene on chord quality. These properties, in turn, are "abstractions of certain aspects or facets" of chord quality. Each such theory provides constructive principles for genera that lump together pcset-classes sharing such properties, and therefore makes an implicit intuitive claim about the aspects and facets of chord quality even if the explicit language is carefully nominal. The goal of the present work is to justify those implicit intuitive claims from the top down, without attempting to ground the theory in the quicksand of intuition; rather, the argument will have its foundations in the usual mathematical and nominal characterization of pcset theory, and will proceed by means of theoretical unification.

There are certain highly characteristic pcset-classes so ubiquitous as to have familiar names in relatively widespread use: chromatic clusters; quartal or quintal chords; Perle's interval cycles (whole-tone scales, augmented triads, and diminished-seventh chords) and combinations of these (Messiaen's modes of limited transposition). Each of these plays a special role in various kinds of pcset theory; each is associated with a unique type of intervallic profile, and each has a relatively limited repertoire of abstract subsets and supersets. Taxonomic theories of atonal harmony typically place such pcset-classes in different genera (which, in turn, are often characterized with reference to those pcset-classes), and similarity relations generally agree that these landmarks are all distant from one another. Even treatments of "twentieth-century harmony" that do not participate in the pcset-theoretic tradition (e.g., Hanson, 1960; Persichetti, 1961, and the last few chapters of many tonal-harmony books) end up focusing on these types of chords and sonorities.

At the level of analytical discourse, we are accustomed to hearing about Skryabin's "mystic chord" as a close relative of the whole-tone scale and diatonic collections (Callender, 1998), of harmonies in Stravinsky's *Sacre* as being nearly octatonic (van den Toorn, 1987, esp. pp. 207–11). Neo-Riemannian theory has opened our eyes to the close relationship between (on the one hand) major and minor triads and (on the other) the augmented triad and hexatonic scale (Cohn, 2000). Boretz (1972) described relations among diatonic seventh chords in the *Tristan* prelude with respect to the structural properties of the diminished-seventh chord.

All of these familiar pitch-class structures are landmarks in the geography of harmonic space — as such, they emerge prominently in the maps of many mapmakers. While the mapmakers may disagree over principles of cartography, they are all mapping the same terrain. We will investigate the extent to which that terrain can be abstracted from the maps at our disposal, attempting to recover some common ground.

§ **1.1 Theorizing about categories.**

By comparing arbitrary chords to a limited number of "highly characteristic" types, we engage implicitly in the same sort of categorization that we do at the most basic levels of cognition. Cognitive scientists today generally agree that we mentally structure categories in terms of **prototypes,** central members of a category whose other members resemble the prototype(s) to a certain degree. (Instructed to think of a chair, you probably would not instantly come up with a beanbag or a porch swing — but those would be more likely than a white tiger or a candy bar.) In a standard work on the subject, Lakoff (1987, particularly pp. 16–57) gives a brief survey of modern thinking about cognitive categorization, some of the highlights of which will be reviewed here.

Lakoff's intent is to problematize what he calls the classical theory of categories, which holds that a category has sharp boundaries determined by some combination of necessary and sufficient conditions. This is the sort of category that classical sets model, of course. Lakoff begins his discussion with Wittgenstein (1953), attributing to him several revolutionary ideas about categories. For Wittgenstein, a category (his well-known example is the category of games) has unclear and extensible boundaries that are not drawn by necessary and sufficient conditions, but by family relationships — similarities of many different kinds and degrees. At the same time, Wittgenstein allows that one can distinguish between good and bad examples of a category; to return to a previous example, a beanbag is not a good example of the category of chairs, even though it bears family relationships to other members of the category, and in particular to good examples. Lakoff observes that the challenge posed to philosophy by Wittgenstein's conception is that the classical theory of categories *qua* sets has no room for good and bad examples, and identifies Zadeh's (1965) theory of fuzzy sets as a first formal attempt to deal with that challenge (see also Quinn, 1997, 2001).

A great body of empirical work by social scientists in the 1960s and 1970s established Wittgenstein's model as a useful point of departure for modeling certain cross-cultural features of human thought. One of the most influential studies was by Berlin and Kay (1969), who presented convincing evidence that, in Lakoff's words,

> Basic color terms name basic color *categories*, whose central members are the same universally. For example, there is always a psychologically real category RED, with focal red as the best, or "purest," example. . . . Languages form a hierarchy based on the number of basic color terms they have and the color categories those terms refer to. . . .
>
> black, white
> red
> yellow, blue, green
> brown
> purple, pink, orange, gray (p. 25)

That is, languages having fewer color terms than those listed invariably have a term low on the list only if they have all of the terms higher on the list; no language has a word for *brown* without also having a term for *red*. Moreover, there was evidence to suggest that certain "focal" colors were better examples, cross-culturally, of these universal color categories than others. Subsequent work by neuroscientists on the perception of color in macaques led to a the development by Kay and McDaniel (1978) of a hierarchical model of color categorization based on Zadeh's fuzzy sets. The model, which was based on the sensitivity of retinal cells to specific wavelengths, successfully accounted for large parts of the linguistic hierarchy discovered by Kay and Berlin, especially as far as the focal colors were concerned.

These studies (among studies of other kinds of categories that Lakoff describes) provided an empirical basis for Wittgenstein's characterization of the general structure of categories as conceptual entities. What had not yet been answered was the question of what sorts of categories we tend to form in response to our observation of things in the world. Lakoff details a number of anthropological studies, undertaken by the aforementioned Berlin and his associates, of the ways in which members of different cultures categorize plants and animals and compares them with the taxonomy laid out by Linnaeus, concluding that "the genus was established as that level of biological discontinuity at which human beings could most easily perceive, agree on, learn, remember, and name the discontinuities.... Berlin found that there is a close fit at this level between the categories of Linnaean biology and basic-level categories in folk biology" (p. 35). Findings such as this — which suggest that for any taxonomic hierarchy, there is a psychologically *basic level* of categories akin to biological genera — were synthesized by Eleanor Rosch into what has now become the standard view of categories: that we organize categories (which have prototypes, or focal elements) into taxonomic hierarchies with a basic level. Paraphrasing from an important article by Rosch and several of her collaborators (1976), Lakoff (p. 46) characterizes the basic level as, among others,

- The highest level at which category members have similarly perceived overall shapes.
- The highest level at which a single mental image can reflect the entire category.
- The level with the most commonly used labels for category members.
- The first level to enter the lexicon for a language.
- The level at which terms are used in neutral contexts.
- The level at which most of our knowledge is organized.

Lakoff concludes by observing (following Rosch) that we must be wary of giving prototypes too important a role in any theory of mental representation for categories:

"Prototype effects, that is, asymmetries among category members such as goodness-of-example judgments, are superficial phenomena which may have many sources" (p. 56). This warning cuts two ways. One the one hand, one must not reason, from the apparent naturalness (or, more neutrally, near-universality) of basic-level categories and their prototypes, to the conclusion that things in the world have inherent properties that sort them into those categories and determine whether or not they are prototypes; see the discussion of the "Myth of Intension" in Quinn (2001). Categories are products of the mind; it is a commonplace among biologists that there are not necessarily "natural kinds" corresponding to taxonomic divisions. On the other hand, one should not take the prototype/basic-level theory of categories to mean that peripheral members of some category are conceptualized with reference to the prototypes of the category — only that prototypes tend to stand at the confluence of the different family relations that constitute the category in the first place; see the discussion of the "Myth of Staggering Complexity" in Quinn (2001).

This is a point of departure for Lakoff, and it is where we leave his particular approach to categories aside, in order to return to the geography of chord quality. Lakoff is primarily concerned with high-level inquiry into the workings of language and the "embodied mind," and the notion that categorization is the very substrate of conceptualization. Zbikowski (2002) provides a rich discussion of the musical issues that fall out of this notion, showing that musical understanding emerges from the interaction of conceptual entities that have a feedback relationship to categorical or taxonomic knowledge — at once grounded in categories shared (as style knowledge) among members of a musical community, and constitutive of these same categories. Work of this kind cannot proceed without a deep theoretical understanding of the nature of the categories themselves, and it is clear that we lack such an understanding for the harmonic categories generally treated under the problematic headings of "post-tonal" or "twentieth-century" harmony. Our motivation is to provide such an understanding, and with it a foundation for higher-level conceptualizations of harmony.

Lakoff's particular manner of framing the issue of categorization (and Zbikowski's success in generating from it a theory of musical understanding) grounds the basic assumptions of our inquiry. There is an evident consensus among theorists that there are basic-level categories of harmony that are hierarchically superior to those of the **chord** and the **species**, which terms will henceforth replace the clumsier formulations *pcset* and *pcset-class*, respectively. (We will continue to call these basic-level categories **genera**, intending to refer not to the specific theories of Forte (1988) and Parks (1989), but to any theory of harmonic relationships that transcends the species level.) Moreover, overwhelming implicit and explicit evidence from the theoretical literature suggests that certain characteristic chord species (including those mentioned at the end of the introduction to this chapter) are prototypes of genera that have many of the features Rosch attributes to basic-level categories.

§ **1.2 The intervallic approach to chord quality.**

The *locus classicus* of chord quality is often taken to be the interval-class vector; Straus, for example, observes that "the quality of a sonority can be roughly summarized by listing all the intervals it contains" (2000, p. 10). Howard Hanson seems to have been the first to use this principle as the basis for a complete and rigorous pcset classification system:

> In a broader sense, the combinations of tones in our system of equal temperament — whether such sounds consist of two tones or many — tend to group themselves into sounds which have a preponderance of one of these [interval classes]. In other words, most sonorities fall into one of the six great categories: perfect-fifth *types*, major-third *types*, minor-third *types*, and so on. (1960, p. 28)

A large folding chart provided with Hanson's book enumerates the harmonic species and classifies them into his seven genera (the six he describes, plus one catchall category for pcset-classes without a predominant interval class).

1.2.1 Prototypes. Hanson clearly delineates a set of prototypes for his categories — these are what he calls the **projections** of the six interval classes. For ic 1 and ic 5, the projections are easily defined as those species whose exemplars are contiguous segments of the chromatic scale and circle of fifths, respectively. In projecting the other interval-classes, Hanson runs into the problem of the interval cycle: "We have observed," he writes,

> that there are only two intervals which can be projected consistently through the twelve tones, the perfect fifth and the minor second. The major second may be projected through a six-tone series and then must resort to the interjection of a "foreign" tone to continue the projection, while the minor third can be projected in pure form through only four tones.
>
> We come now to the major third, which can be projected only to three tones. (1960, p. 123)

Hanson's rather ingenious solution to the problem concerns the addition of a "foreign tone" into the projection, which introduces a pitch-class outside of the just-completed interval cycle and provides a starting point for the continuation of the projection. Invariably, his "foreign tone" is a fifth above the starting point, although a shift of a semitone would work just as well.

It is relatively easy to describe his procedure for generating chordal prototypes as an algorithm, although he does not do so explicitly (for unclear reasons, he uses a different procedure with the tritone, although he ends up with identical results). Figure 1.1 describes the procedure for generating a chord p that is the species prime form of a d-note projection of interval-class i. The internal variable n stands for notes added to the chord. The layout of the flowchart clearly shows that Hanson's procedure has the structure of a nested interval cycle — an "inner cycle" of the interval-class i being projected, and an "outer cycle" (for Hanson, a 7-cycle, and for us, a 1-cycle) that generates foreign tones in order to make available fresh transpositions of the i-cycle.

Figure 1.2 displays the species prime form of all chords generated by Hanson's procedure, using both the traditional Forte nomenclature and clockface diagrams. (The

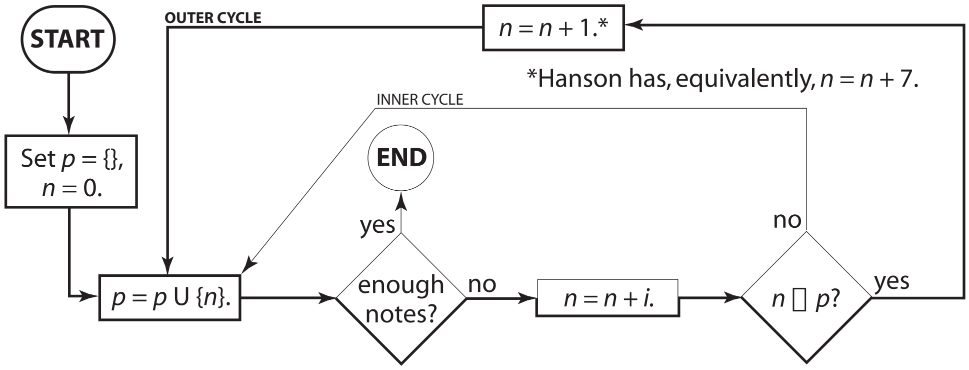


FIGURE 1.1. *Algorithmic description of Hanson's projection procedure.*

clockface diagrams include some additional graphical apparatus that will be explained shortly.) "Trivial" species corresponding to the null pcset, the aggregate, and singletons and their complements are included as well.

As Hanson himself observes, his choice of the perfect fifth as the interval that generates the "foreign tone" is essentially arbitrary. He allows that other solutions will work in individual cases (such as minor thirds in the ic-4 projection), but that only foreign-tone cycles of fifths or semitones will work in all cases. In this connection it is interesting to contemplate Figure 1.3, reproduced from Headlam (1996), which duplicates Hanson's approach to projection, but with foreign tones introduced along a cycle of semitones rather than fifths, as we have done in Figure 1.1. In fact, our algorithm, when allowed to run all the way through the aggregate, generates precisely the same notes as appear in each line of Headlam's figure.

Viewed as a complete system, Hanson's projections have five properties that make them particularly attractive as a set of prototypes for harmonic genera:

The Unique-Prototype Property (UPP): Each of the six genera has one, and only one, prototypical species of any given cardinality. This feature makes Hanson's system quite tidy, but it is not necessarily a requirement of a conceptually robust taxonomy.

The Unique-Genus Property (UGP): Each of Hanson's projections is a prototype of one, and only one, genus. In contrast to the UPP, this

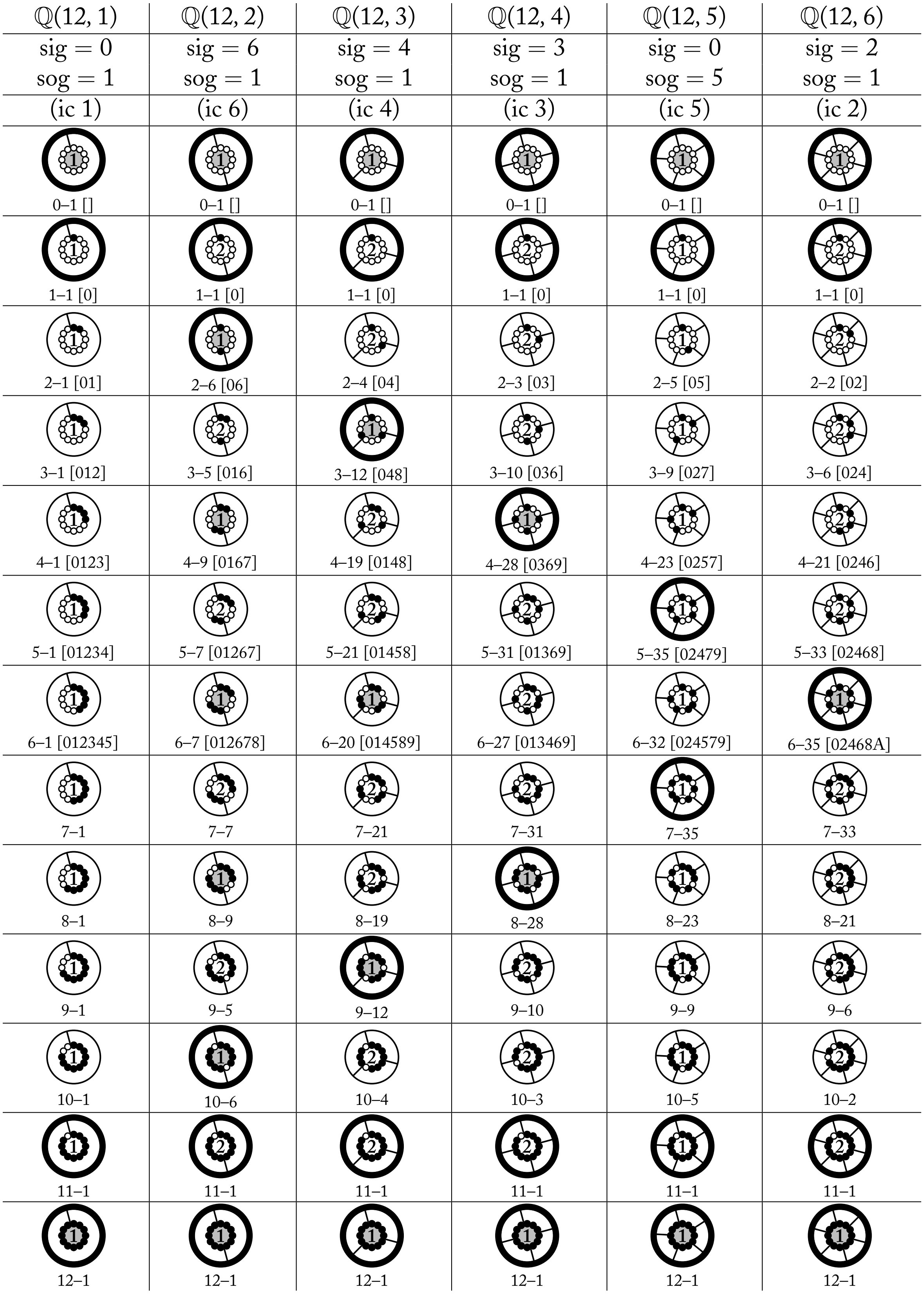


FIGURE 1.2. *Hanson's projections; tentative prototypes of the qualitative genera* $\mathbb{Q}(12, n)$.

FIGURE 1.3. *Figure 1.1 from Headlam (1996).*

would seem to be necessary for conceptual robustness; after all, a taxonomy that cannot unambiguously classify its prototypes necessarily cannot be trusted to classify anything else.

The Intrageneric Inclusion Property (IIP): Each generic prototype (abstractly) includes all smaller prototypes of the same genus. In the broader context of manifold theories of harmony, many of which privilege inclusion relations (even tonal theory does this in several important ways), this is a desirable feature.

The Prototype-Complementation Property (PCP): The complement of any generic prototype is another prototype of the same genus. Many aspects of pcset theory (especially those connected with Forte's work) and twelve-tone theory are concerned with complementation. Even tonal theory depends on complementation to some extent, when it comes to distinguishing harmonic functions in terms of quasi-complementary relationships *within* diatonic collections.

The Prototype-Familiarity Property (PFP): Many of these prototypes are chord species of the sort discussed in the introduction to this chapter, species that have familiar names and important roles in a wide range

> of analytical applications. This is an important part of what makes harmonic genera function as basic-level categories. In Figure 1.2, some of the clockface diagrams have thicker circles surrounding them, and some have shaded interiors. The former are maximally even, and the latter are transpositionally invariant; all are familiar.

Hanson characterizes the constitutive feature of his genera as the "preponderance" of some interval class. This does not tell the whole story, though, since — to cite one example — while the whole-tone collection (6–35) has exactly as many instances of ic 4 as the hexatonic collection (6–20), only the latter is a prototype of the ic-4 genus, thanks to some hand-waving on Hanson's part as concerns the foreign-tone technique: in a footnote, he discounts foreign-tone generators for the ic-4 cycle that would result in whole-tone formations on the grounds that whole-tone formations properly belong in the family of ic-2 projections. No detailed and general treatment of the issue is to be found in Hanson's book.

Notwithstanding Hanson's failure to treat this issue with more rigor, his projections are, in many ways, ideal prototypes of intervallically constituted genera, since each projection is what Eriksson (1986) calls a **maxpoint**, a chord species "containing the maximum number for its size of at least one interval class" (p. 96). The maxpoints of ics 1 and 5 correspond one-to-one with Hanson's projections; maxpoints of the other ics are tabulated in Figure 1.4. Eriksson's treatment of maxpoints occurs in the context of his own development of a taxonomic theory for harmonic species, and the maxpoints are his initial candidates for generic prototypes (he refines the system later in his paper). Immediately apparent from the proliferation of chord species in the ic-4 and ic-6 columns of Figure 1.4 is the fact that as a set of prototypes, maxpoints do not have UPP. Nor, for that matter, do they have UGP — species marked with a superscript *d* are maxpoints of more than one interval. All of the offenders are either projections of ic 3 that also happen to have the maximum number of tritones, or projections of ic 2 that also happen to have the maximum number of both major thirds and tritones.

card.	ic 2	ic 3	ic 4	ic 6
2	2–2 [02]^{p}	2–3 [03]^{p}	2–4 [04]^{p}	2–6 [06]^{p}
3	3–6 [024]^{p}	3–10 [036]^{pd}	3–12 [048]^{p}	3–5 [016]^{p} 3–8 [026] 3–10 [036]^{d}
4	4–21 [0246]^{p}	4–28 [0369]^{pd}	4–19 [0148]^{p} 4–24 [0248]	4–9 [0167]^{p} 4–25 [0268] 4–28 [0369]^{d}
5	5–33 [02468]^{pd}	5–31 [01369]^{pd}	5–21 [01458]^{p} 5–33 [02468]^{d}	5–7 [01267]^{p} 5–15 [01268] 5–19 [01367] 5–28 [02368] 5–31 [01369]^{d} 5–33 [02468]^{d}
6	6–35 [02468A]^{pd}	6–27 [013469]^{p}	6–20 [014589]^{p} 6–35 [02468A]^{d}	6–7 [012678]^{p} 6–30 [013679] 6–35 [02468A]^{d}

(^{p}) indicates a maxpoint that is also a Hanson projection; (^{d}) indicates a “duplicate” maxpoint of several ics.

FIGURE 1.4. *Eriksson’s maxpoints.*

One way to improve the situation would be to change the definition of a maxpoint. Isolating certain special maxpoints (call them **supermaxpoints**) that not only maximize some ic, but also have strictly more occurrences of that ic than any other chord species of the same cardinality, reinstates a weak form of UPP. Rather than having multiple prototypes of the same cardinality in certain genera, we now have no prototypes for certain cardinalities in the genera associated with ics 4 and 6; otherwise the supermaxpoints are coextensive with Hanson’s projections (see Figure 1.5). At the same time, this maneuver solves the problem of UGP, since each of the multiply affiliated maxpoints marked with a *d* in Figure 1.4 is a supermaxpoint of exactly one interval class. And, in fact, the supermaxpoints are all also projections.

Noting problems akin to those that could be circumvented by isolating the supermaxpoints, and after an extensive discussion of abstract inclusion relations among maxpoints (clearly related to IIP), Eriksson eventually abandons the idea of using chord species as prototypes at all. Rather, his “model” for each of his seven genera (he calls

card.	ic 1	ic 2	ic 3	ic 4	ic 5	ic 6
2	2–1 $[01]^{p}$	2–2 $[02]^{p}$	2–3 $[03]^{p}$	2–4 $[04]^{p}$	2–5 $[05]^{p}$	2–6 $[06]^{p}$
3	3–1 $[012]^{p}$	3–6 $[024]^{p}$	3–10 $[036]^{p}$	3–12 $[048]^{p}$	3–9 $[027]^{p}$	
4	4–1 $[0123]^{p}$	4–21 $[0246]^{p}$	4–28 $[0369]^{p}$		4–23 $[0257]^{p}$	
5	5–1 $[01234]^{p}$	5–33 $[02468]^{p}$	5–31 $[01369]^{p}$		5–35 $[02479]^{p}$	
6	6–1 $[012345]^{p}$	6–35 $[02468A]^{p}$	6–27 $[013469]^{p}$		6–32 $[024579]^{p}$	

FIGURE 1.5. *The "supermaxpoints."*

	I	II	III	IV	V	VI	VII*
typical ics	1	2, 4, 6	3, 6	4	5	6	2
↓	2 3, 4			1, 3, 5	2 3, 4	1, 5	1, 5, 3, 4
atypical ics	5, 6	1, 3, 5	1, 2, 4, 5	2, 6	1, 6	2, 3, 4	6

*Only M-invariant chord species may belong to this genus.

FIGURE 1.6. *Models of relative ic multiplicity in Eriksson's seven "regions" (genera).*

them *regions*) is a partial ordering of interval classes, ordered according to their frequency of occurrence in a given species of chord. Figure 1.6, adapted from his Example 6, lists the models for the seven genera he eventually asserts. The highly suggestive (and innovative) move from intervallic maxpoints to regional models is not without problems — for example, the Petrushka chord 6–30 [013679] is a maxpoint of ic 6, but its interval vector, ⟨2, 2, 4, 0, 2, 3⟩, resembles the genus III model of high ic-3 and -6 content much more than it does the genus VI model, which specifically provides for low ic-3 content.

1.2.2 Intrageneric affinities. In related work on a similarity-oriented theory, Michael Buchler (2001) suggests that in order to compare two chord species on the basis of their ic content, it is beneficial to contextualize the interval-class vector, as he puts it, by means of "tools that take account of what is minimally and maximally possible in a given cardinality" (p. 264), judging ic content relative to such possibilities. Buchler calls the relativized measure of ic content the *degree of saturation*. A maxpoint for some ic is, in Buchler's terms, fully saturated with that ic. Setting aside for the moment the considerations that lead Eriksson to supplement the theory of maxpoints with ic-vector "models" in asserting his generic prototypes, we observe that Buchler's generalization of maxpoints into degrees of saturation provides a fuzzification of the prototype idea. A prototype is a very good example of a category; other members of a category may be ranked in terms of their own goodness-of-example as well. We will use the term **intrageneric affinity** to refer to this goodness-of-example relationship.

To the extent that interval content determines the intrageneric affinities of intervallically constituted genera, one way to figure the affinity (degree of membership) of a chord in such a genus might be to determine the degree to which the pcset-class is saturated with the relevant ic (Buchler's footnote 11 may be taken to suggest something along these lines). Yet this sort of classification system would have as generic prototypes *all* maxpoint pcsets, and thus inherit the UPP- and UGP-related problems discussed above — the same problems that Hanson and Eriksson seek to avoid, each in his own way. Recognizing this problem, Buchler defines what he calls the "maximal cyclic fragmentation condition" (p. 270), which is met by all and only the maxpoints singled out as prototypes under Hanson's system of projections (indeed, Buchler's condition is precisely that used implicitly by Hanson). Buchler's work is the background for a similarity relation, though, and not a classification scheme, and he does not offer any suggestions as to how one might go about fuzzifying the "maximal cyclic fragmentation condition" in a way that would model intrageneric affinities for a system of genera constituted by this specialized notion of interval-class saturation. For that matter, neither Hanson nor Eriksson makes any finer distinction between members of a genus than between prototypes and nonprototypes.

If the intrageneric affinities of a genus are the degrees to which various chord species exemplify the genus, and if the best examples of a genus are its prototypes, then a natural way to measure them with extant tools of pcset theory — and to provide the "finer distinctions" we do not get from Buchler, Hanson, or Erikkson — is to consider the similarity of a chord to generic prototypes using fuzzy similarity relations that are based on interval content, such as Morris's SIM and ASIM relations (1979), Isaacson's IcVSIM (1990) and his various ISIM$_n$ relations (1996), and Scott and Isaacson's

ANGLE (1998). Scott and Isaacson provide a technical overview of the mathematical connections among these relations, but all essentially measure the degree to which two ic vectors have the same profile — in stark contrast to the "original" similarity relations from Forte (1973), which neither come in degrees nor treat the ic vector as the sort of thing that can have a shape, instead focusing on the yes-or-no question of whether corresponding entries in two ic vectors are equal.

Fuzzy similarity relations do not directly suggest a system of prototypes, although they do suggest genera (see Quinn, 2001). The discussions we have undertaken so far, in connection with the intensional characterization of the aforementioned similarity relations as being oriented toward interval content, suggest that Hanson's system of prototypes, or something like it, could form the basis of a generic taxonomy, with similarity relations predicting (or modeling, if you prefer) the intrageneric affinities of the genera. A more purely extensional approach might derive from the technique of multidimensional scaling, which treats dissimilarity measurements among objects as distances in a multidimensional space, then finds a distribution of the objects in the space that provides a good fit with the data. Highly dissimilar objects will be far apart, and highly similar objects will be close together. The application of multidimensional scaling to data from chordal similarity relations tends to produce distributions in which certain chords are at the "edges" of the distribution — thereby being as far away from one another as possible — and these chords tend to be the prototypes we have been discussing. Cognitively oriented work on multidimensional scaling of chord similarity data has been conducted by Mavromatis and Williamson (1999), Samplaski (2000), and Kuusi (2001).

Another approach might proceed from Eriksson's prototypical models of ic-vector shape. Block and Douthett (1994), who do not cite Eriksson's article, present a related, but more general approach (see, e.g., p. 22: "a composer may wish to find a family of sets that have certain intervals suppressed or eliminated and at the same time have others emphasized"). They develop a general structure (a "weight vector") that corresponds to such situations, and a procedure derived from vector algebra for determining how well a particular ic vector exemplifies this weighting. Many of the examples in their article consist of a table of chord species arranged according to goodness-of-example, and although they do not take the step of isolating prototypical ic-vector shapes — this was Eriksson's major achievement — it is the case that the ten such examples they adduce are headed up by chord species corresponding to Hanson's projections.

While any particular instance of these two approaches will likely produce similar taxonomies (see Quinn, 2001), such an ad-hoc model would have pragmatic rather than explanatory value, and not much pragmatic value at that: it would be useful for answering questions that only straw men are asking ("how well is genus *g* exemplified by species *s*?") without shedding any light on interesting, abstract questions about why so many superficially different theories converge at the basic level of chord quality. If we seek more interesting questions, rather than answers to less interesting ones, we should be highly suspicious of any such ad-hoc approach to answering questions about intrageneric affinities.

Without committing to any particular approach yet, we will use the term **Q-space** to describe an idealized spatial distribution of chord species, especially one that reflects the intrageneric affinities of the qualitative genera we have been describing. (The term reflects a scheme for naming qualitative genera with labels such as $\mathbb{Q}(12, 1)$, used in Figure 1.2 and explained in § 2.3, *infra*.) The prototypes of each qualitative genus lie in a particular "population center" of Q-space, and the spatial position of an arbitrary chords reflects, by its distance from these various remote regions, its affinity to each of the qualitative genera. Our ultimate goal is a theory of chord quality that describes Q-space more specifically, and that makes the prediction that multidimensional scaling of data generated by any of the particular procedures described above will produce spatial distributions that converge on Q-space.

Eriksson's graphic depiction of his "regions," reproduced here as Figure 1.7, is particularly suggestive in this regard — we may interpret his genera somewhat literally as regions of Q-space. A peculiarity of Eriksson's layout, and of his system of genera, is the relationship of his genera I, V, and VII. In the graphic layout, he suggests that

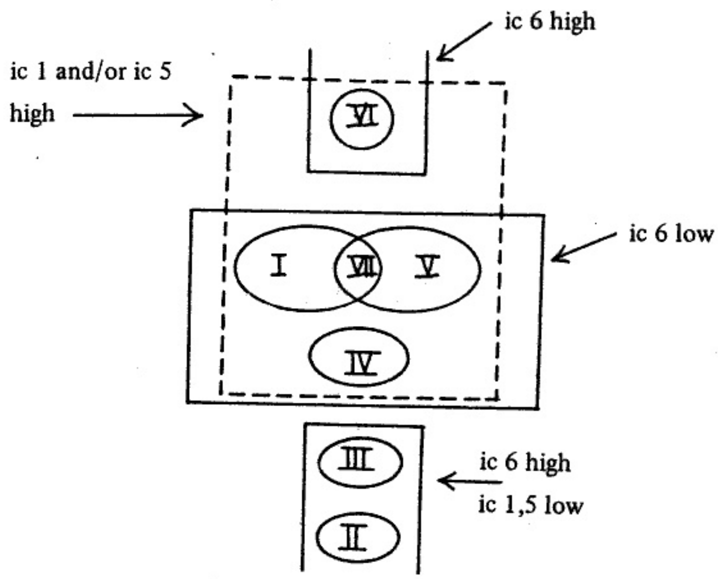


FIGURE 1.7. *Eriksson's graphic representation of his genera.*

VII is constituted by the overlap of peripheral parts of I and V; and in his description of the intervallic model of genus VII he includes the caveat that such chord species must contain equal numbers of ic 1 and ic 5, which is tantamount to asserting that they must be invariant under M, the circle-of-fifths transform. These points, which we will revisit a bit later (§ 1.3.2), throw into question the status of his genus VII as an independent "population center," since he seems to characterize it more as disputed territory between the regions of genera I and V. Under this reading, we are left with (more or less) the usual six intervallically constituted genera.

1.2.3 Intergeneric affinities. We have been imagining a fuzzy taxonomy for chord species in which there are six genera roughly corresponding to the six interval classes. Each genus is endowed with a structure described by its intrageneric affinities, which depend on the notion of generic prototypes (best examples of genera) and the the idea that the similarity of an arbitrary chord to the prototypes of a genus is equivalent to the goodness-of-example of the chord to the genus. We now consider the question of how genera are related to one another, or how one might go about drawing analogies between genera. Our case study will involve the genera associated with ics 1 and 5.

Two basic observations will get us going. First: any prototype of one of these two genera, when subjected to the M_5 (circle-of-fourths) or M_7 (circle-of-fifths) transforms, yields a prototype of the other genus; this is the case irrespective of whether one chooses Hanson's projections, Eriksson's maxpoints or ic-vector models, our supermaxpoints, or Buchler's saturated chords to serve as prototypes. Second: suppose we have two chords, and their degree of similarity is *s* (as measured by any of the similarity relations mentioned above); transform both chords by M (by which we mean "either M_5 or M_7"), and the degree of similarity between the transformed chords is also *s*. Choosing an example at random, let *p* be an examplar of the species 4–27 [0258], and let *q* be an exemplar of the species 6–13 [013467]. Buchler's SATSIM reports their degree of similarity as 0.261, and Scott and Isaacson's ANGLE gives 0.117. Transforming *p* by M yields an exemplar of 4–12 [0236]; *q* becomes an exemplar of 6–50 [014679] under either transformation. Asked about the similarity of these latter two chords, SATSIM gives 0.261 again, and ANGLE gives 0.117 again.

We will not attempt to prove here why intervallically constituted similarity relations behave in this way (see Morris, 2001, ch. 4), but only to show that the two observations just made allow us to make an interesting analogy between the two genera in question. Since the intrageneric affinities of the genera are deterimined by similarity relations involving their respective prototypes, and since the prototypes of the two genera are swapped under certain transformations, and since the similarity relations determining intrageneric affinities seem to be invariant under those same transformations, we can draw the following general conclusion about the relationship among these two genera: The degree to which a species (e.g., 6–3 [012356]) exemplifies one genus is the same as the degree to which its M_5-transform (e.g., 6–25 [013568]) exemplifies the other genus. This conclusion pertains to what we will call the **intergeneric affinities** between the two genera in question.

M_5c	0	5	A	3	8	1	6	B	4	9	2	7
c	0	1	2	3	4	5	6	7	8	9	A	B
"M_2c"	0	2	4	6	8	A	0	2	4	6	8	A

FIGURE 1.8. *Multiplication by 2 as an epimorphism.*

In a sense, this use of the word *affinities* is closely allied to the eponymous doctrine of medieval theory, which concerns the relationship of different, but functionally identical, notes in different tetrachords. Each tetrachord has notes called *protus, deuterus, tritus,* and *tetrardus*; these qualitative labels (intragenerically) describe the positions of those particular notes in, say, the *graves* tetrachord. Affinities among the tetrachords are described (intergenerically) by identity among the qualitative names and related, coincidentally, by transpositions through fourths or fifths — the *protus* of the *graves* tetrachord has an affinity to the *protus* of the *finales* tetrachord a fourth higher, and to the *protus* of the *superiores* tetrachord a fifth above that.

The existence of intergeneric affinities between the two genera we have studied raises the more general issue of affinities between any two genera. The M_5 and M_7 operators relate the genera in question because each "expands" members of ic 1 into members of ic 5, and further expands members of ic 5 into members of ic 1, all the while leaving all other intervals invariant (up to interval-class). To generalize this to other genera, we might briefly consider a multiplicative operator that expands members of ic 1 into other intervals, but here we face Hanson's original problem: the interval cycle. Suppose, for example, we invent an operator that transforms every pc c into $2c$ (mod 12). Figure 1.8 contrasts this "M_2" operator with the usual M_5 operator; the problem is that while M_5 and M_7 (like the usual transposition and inversion operators) are one-to-one mappings of the pitch-class universe onto itself, this "M_2" is a many-to-one mapping. Any two pcs separated by a tritone map to the same pc under "M_2," and no pcs map to an odd pc under this operator.

In an unpublished manuscript (2000), Daniel Harrison notes this problem — which is endemic to the generalization of multiplicative operators in the twelve-pc universe — and presents a suggestive workaround. His N operator, which is a replacement for

⋮													
$N^{-3}c$	0	7	3	A	6	2	9	5	1	8	4	B	
$N^{-2}c$	0	3	6	9	1	4	7	A	2	5	8	B	= “M_3”
$N^{-1}c$	0	6	1	7	2	8	3	9	4	A	5	B	= “M_6”
c	0	1	2	3	4	5	6	7	8	9	A	B	
Nc	0	2	4	6	8	A	1	3	5	7	9	B	= “M_2”
N^2c	0	4	8	1	5	9	2	6	A	3	7	B	= “M_4”
N^3c	0	8	5	2	A	7	4	1	9	6	3	B	
⋮													

FIGURE 1.9. *Harrison's N operator.*

“M_2,” multiplies pitch-class integers by slightly more than 2 ($2+1/6$, to be exact), then rounds down to the nearest integer. Figure 1.9 details the mapping. Just at the point where “M_2” starts to repeat itself (pc 6), duplicating the even whole-tone collection, the extra 1/6 in Harrison's multiplier is amplified past the point of being rounded away, and N maps the remaining six pcs to the odd whole-tone collection.

Harrison's solution is suggestive for two reasons. First, when his N is compounded on itself (N^2), it acts as a substitute for “M_4” (also an epimorphism); furthermore, its inverse (N^{-1}) acts as a substitute for “M_6,” and its inverse compounded on itself (N^{-2}) acts as a substitute for “M_3.” In this way, N serves as a generator of a series of one-to-one mappings that simulate all the other types of multiplication we would need to investigate intergeneric affinities in the six-genus taxonomy we have been exploring.

Second, these uses of N also happen to simulate Hanson's method for generating projections of ics 2, 3, 4, and 6 (compare Figure 1.9 with Figure 1.3 and the associated discussion). As a consequence, it happens that transforming certain exemplars of Hanson's ic-1 projections by N or any of its several compounds just mentioned will yield exemplars of other Hanson projections. In particular, the prime-form exemplar of any ic-1 projection (regardless of cardinality) will transform into an exemplar of some other projection. Clearly this suggests the beginning of a quite general way to assert intergeneric affinities among our qualitative genera. Unfortunately, the interrelationship of prototypes is only guaranteed when we are dealing with prime forms; for example, N transforms {01234}, an ic-1 projection, into {02468}, an ic-2 projection; but it also transforms {45678}, another ic-1 projection, into {1358A} — which is not only not an ic-2 projection, but is also an ic-5 projection! The mathematical issue at hand is outside the scope of our present inquiry, but the upshot is that any chord can be transformed into *any other* chord of the same cardinality by some combination of N with the usual transposition and inversion operators.

While N gets us tantalizingly close to a theory of intergeneric affinities, its explosive interaction with the operators that define chord species tightly circumscribe its usefulness. We will revisit the issue of intergeneric affinities in the next section.

§ 1.3 Other approaches to chord quality.

In addition to the intervallic approaches we have studied, theorists have explored other ways of qualitatively relating chords. As we will see, much of the ground we have covered so far is intimately connected to these other approaches, despite superficial differences.

1.3.1 The inclusional approach. The most widespread alternative approach to chord quality involves the ways in which chords include one another and, by extension, in which chord species abstractly include one another. The large folding chart supplied by Hanson (1960) details abstract inclusion relations among species in a manner that clearly and vividly shows that chords tend to belong to the same genera as their subsets and supersets. Nowhere in his book, however, does Hanson explicitly make this observation, although his general approach (which involves extending the idea of projection from intervals to trichords) is so thoroughly shot through with the idea of inclusion that, to misuse Hanson's own words (p. 272), one "may well ask whether any such detailed analysis went on in the mind of the composer as he was writing the passage. The answer is probably, 'consciously—no, subconsciously—yes'."

Forte's treatment of set complexes of the K and Kh types (1973) represent the first explicit and thoroughgoing study of abstract inclusion relations that has a qualitative character. It is useless as a generic taxonomic system, however, because if a pcset-complex is a genus whose nexus is its prototype, it follows that any chord species whatsoever can be a prototype of some genus. Forte's nested complexes, however, show an interesting parallel with the fuzzy generic structures we have been imagining. Each K-complex, as a category, has a single prototype up to complementation; yet we may interpret the Kh-subcomplex about that same prototype as a class of privileged members of the K-complex — better examples of the genus. In Forte's words, the idea of the Kh-subcomples is to supply "additional refinement of the set-complex concept in order to provide significant distinctions among compositional sets."

Much later, Forte (1988) used similar principles to develop a quasi-taxonomic system of what he calls *genera* and *supragenera*. There is still considerable overlap among categories at the generic and suprageneric levels, but taxonomic hierarchicy is more clearly manifest in this theory, since the supragenera wholly include their respective genera. (At the same time, the number of chord species assigned to just one genus, or even just one supragenus, is disappointingly small, and Forte offers no further means of grading goodness-of-example within either level.) Introducing the system, Forte announces that "we posit the intervallic content of pitch-class sets as the fundamental basis of the genera" (p. 188), although this basis is operative only as far as selecting generic prototypes (which are all trichords) is concerned, and from there the constitutive principle is once again inclusion. Complicating the situation somewhat is the fact that, strictly speaking, not all of his prototypes are trichords — some are *pairs* of trichords that have two out of three ics in common. (For more on the relationship between Forte's genera and his set-complexes, see Morris, 1997.)

The conceptual bridge between thinking about interval content and thinking about abstract inclusion relations was first solidly erected by Lewin (1977), who, observing that an interval-class is simply a species of two-note chord, suggested that the qualitative utility of thinking about interval content might be extended to the consideration of subset content generally. Not long thereafter, responding both to Morris's development of the first graded similarity relation, the ic-based SIM, and to Rahn's subset-based TMEMB, Lewin positioned his own REL as a conceptual generalization of both (Morris, 1979; Rahn, 1980; Lewin, 1979).

Lewin's observations, and other lines of thought that originate with them, may help to explain why the interval-based approach to chord quality converges with the subset-based approaches. On the one hand, similarity relations such as TMEMB and REL that (more or less) simply count common subsets, as well as relations like Castrén's fantastically complicated RECREL (1994), which compares the subset *structures* of chord species rather than just the subsets, produce results that fall in line with those produced by ic-based similarity relations when glimpsed from an extensional, taxonomic point of view (Quinn, 2001). On the other hand, while the prototypes of Forte's system of genera are technically trichords or pairs of trichords, he frequently describes the genera in "very informal descriptive terms so that the genera might seem more accessible and familiar to the reader" (Forte, 1988, p. 200). These terms include *whole-tone, diminished, augmented, chroma*[tic], and *dia*[tonic], which vividly recall the sorts of prototypes that arise under strictly intervallic approaches. In the related generic system of Parks (1989), similarly "accessible and familiar" terms arise. They are "informal" only because none of the builders of formal chord-quality models seems to have been willing to take them seriously enough to seek the appropriate formalizations instead of offering appeals to intuition, which seem rather vacuous in the otherwise highly rigorous context of pcset theory.

There are three good reasons to take these accessible labels seriously. The first has to do with prototypes — the prototypes of a qualitative taxonomy constituted by inclusion relations would seem to be precisely the same as those that fall out of an intervallic approach. In particular, we have observed that Hanson's prototypes have IIP and PCP, which amounts to saying that the prototypes of any qualitative genus are Kh-related to each other. We have also observed that they have PFP, which establishes a strong conceptual connection to Forte's and Parks's "accessible and familiar" names for their own genera.

The second concerns intrageneric affinities, the modeling of which we have been delegating to fuzzy similarity relations. It having been established that inclusional similarity relations end up producing largely the same results as intervallic similarity relations, there is good reason to suppose that they would agree as to the intrageneric affinities of genera constituted by our working set of familiar intervallic-cum-inclusional prototypes.

The third involves intergeneric affinities. We have yet to get terribly far with this concept, but our observations concerning the M_5 and M_7 operations, which establish affinities between $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$ — now that we are no longer discussing interval content *per se*, we will henceforth refer to qualitative genera by these symbols, which were silently introduced in Figure 1.2 — carry over to the inclusional realm as well. Suppose we have a chord p that includes a chord q; it follows immediately that M_5p includes M_5q. From there it is a short journey to the conclusion that whatever an inclusional similarity relation says about species s and t will be the same as what it says about M_5s and M_5t, something that is borne out by all of the inclusional similarity relations mentioned.

Most existing approaches to qualitative chord taxonomy belong to either the intervallic or inclusional approaches. We have seen how these two types may be linked both technically and conceptually, and how they tend to create similar sorts of genera. This, perhaps, raises the question of which approach is "right." On the face of it, this is rather a silly question, which seems to invite intensional discourse of the sort I have argued fervently against elsewhere. Hanson, for instance, *says* his projections are intervallically conceived, but that does not disallow us from asserting, as we have, that they also have a strong inclusional aspect, and that they seem to reflect maximal evenness and transpositional invariance in some way as well. At the same time, I am going to argue later on that neither the intervallic nor the inclusional approach is "right," and so I do not wish to throw out the question as intensional and therefore trivial. Rather, the question should be viewed as theoretical and therefore explanatory in nature — the theory of chord quality we are developing seems to transcend the distinction between intervallic and inclusional approaches, largely because the intervallic and inclusional aspects of the generic prototypes and affinities converge; neither is necessary or sufficient to explain the other. Furthermore, there are a few loose ends concerning maximal evenness and transpositional invariance (which seem to be contributing factors to the all-important PFP), and considerably more loose ends concerning intergeneric affinities, that we are as yet at a loss to explain.

Our survey will continue and conclude with a study of two sharply distinctive approaches to the construction of genera that will take us away from the intervallic and inclusional approches, and advance us toward an understanding of said loose ends. These two approaches are truly *sui generis*, so to speak, in terms of both the constitution of genera and the nature of the genera that are thereby constituted.

1.3.2 Morris's algebraic approach. In the tradition of mainline pcset theory, Morris (1982, 2001) adheres strictly to what Lakoff characterizes as the "classical view" of categories — that the extension of a category is a set, and that the intension of a category is a collection of necessary and sufficient conditions. Morris begins with a survey of the common approaches to gathering chords into species, referring to each such approach as a *set-group system*. He cites four, describing the intensional conditions for species in each system; to paraphrase:

- Under $SG(v)$, two chords belong to the same species if and only if they have the same interval content.
- Under $SG(1)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition.
- Under $SG(2)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion.

- Under $SG(3)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion and/or the circle-of-fifths transform M.

Morris uses the term *set-group* in a highly general sense; because he does not assert anything like a basic level of classification, there is no way to make a distinction of kind between species and genera. Yet the taxonomic instinct looms large in his theory of set-group systems. Much of this theory is concerned with the ways in which one set-group system can be hierarchically related to another, in the hard-and-fast manner of Linnaean species, genera, familiæ, ordines, and so forth up the taxonomic ranks.

Generalizing Lakoff's terminology, we might assert a classical view of taxonomy that can be characterized inductively: each taxonomic rank is a classical system of categories, and any two things classified in the same category at a given rank are necessarily classified in the same category at all higher (coarser) ranks. Morris's discussion of the four set-group systems just listed strongly implies a classical view of taxonomy; much of his initial discussion amounts to the observation that $SG(1)$, $SG(2)$, and $SG(3)$ may be successive ranks of a classical taxonomy. His subsequent development of an array of additional set-group systems seems to be motivated by frustration that $SG(v)$ cannot be worked into a classical taxonomy with the other three. In particular, $SG(v)$ is taxonomically superior to both $SG(1)$ and $SG(2)$, but is taxonomically neither superior nor inferior to $SG(3)$. (Counterexamples are 5–12 [01356] and 5–36 [01247], which are not M-related but do have the same interval content; and ics 1 and 5 themselves, which are M-related but clearly do not have the same interval content.) Morris initially suggests a bit of fudging: if we were to decree ic 1 and ic 5 to be identical, redefining "interval content" accordingly, the problem would go away, with $SG(v)$ taking its place in the hierarchy just above $SG(3)$.

Assuming that by "species" we mean the taxa of $SG(2)$, which are the just the familiar pcset-classes of Forte (1973), we may choose either $SG(3)$ or any of its superior set-group systems to function as what we have been uniformly calling genera. For reasons that will be immediately evident, let us distinguish between the **algebraic genera** of Morris's

	ic-1 genus	ic-5 genus
prototype	5–1 [01234]	5–35 [02479]
↓	5–2 [01235]	5–23 [02357]
↓	5–3 [01245]	5–27 [01358]
↓	5–9 [01246]	5–24 [01357]
distant	5–33 [02468]	5–33 [02468]

FIGURE 1.10. *Morris's algebraic* $SG(3)$ *genera (rows) versus qualitative genera (columns).*

set-group taxonomy and the **qualitative genera** that have been our main topic thus far. Even with $SG(3)$ itself, we see something completely different happening in Morris's taxonomy than with all of the other approaches we have discussed to this point. Recall that all of those systems (implicitly or explicitly) include separate qualitative genera for ic 1–type and ic 5–type chord species, the prototypes of which are connected segments of the chromatic scale and the circle of fifths, respectively; then note that by assigning M-related chord species to the same algebraic genus, the $SG(3)$ system asserts algebraic-generic equivalence between pairs of qualitative prototypes. Any $\mathbb{Q}(12, 1)$ prototype is the M-transform of the $\mathbb{Q}(12, 5)$ prototype of the same cardinality, and vice versa.

Consider now the chord species displayed in Figure 1.10. They are arranged into two columns that correspond to the fuzzy qualitative genera $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, respectively. The top row contains prototypes of each genus, and successively lower rows contain species more distant from the prototypes (most similarity relations will agree on this). At the same time, the two entries in each row are M-related to one another, and consequently belong to the same "genus" of $SG(3)$. What this means is that $SG(3)$ brings together species that play the *same* role in two *different* qualitative genera that are wholly M-related to one another.

What we are dealing with, of course, is the the same phenomenon we discussed earlier in connection with intergeneric affinities between the qualitative genera. Two chords belong to the same *algebraic* genus of $SG(3)$ if and only if there is an intergeneric affinity between them with respect to the two *qualitative* genera in question. In this sense, Morris's algebraic genera are exactly orthogonal to qualitative genera — and in a way that engages an important feature of the system of qualitative genera.

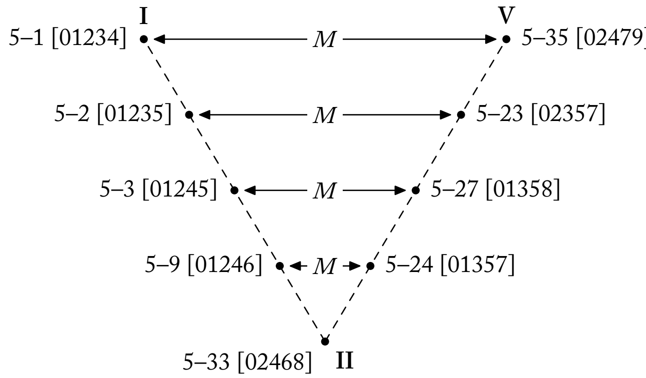


FIGURE 1.11. *M_5 structures certain intergeneric affinities; it also preserves certain intrageneric affinities.*

Referring back to Eriksson's map of his genera (Figure 1.7), and pretending that this is in fact a map of Q-space, let us locate the chord species of Figure 1.10 on the map. The locations of the prototypes 5–1 [01234] and 5–35 [02479] are where Eriksson has marked "I" and "V," respectively. The species 5-33 [02468], in the bottom row of Figure 1.10, is a prototype of the qualitative genus $\mathbb{Q}(12, 6)$, which, because of that genus's association with ic 2, can therefore be located where Eriksson has marked "II." Now draw two straight lines from 5–33: one to each of the two prototypes 5–1 and 5–35. Distribute the other chord species from the two columns of Figure 1.10 evenly along those two lines, and in the same order. The resulting distances are very roughly those that most similarity relations will estimate for all of the chord species involved.

The arrangement just described, and sketched in Figure 1.11, is consistent with the following assertion: The M operation corresponds to a reflection (flip) about a vertical line running through the center of Figure 1.7. This assertion, in turn, is consistent with two important observations. First, as we have already noted, Eriksson's genus VII consists, by fiat, entirely of M-invariant chord species. In his graphic representation, that genus lies between I and V and directly on the axis of reflection corresponding to M. Second, as we have not previously noted, nearly all of the agreed-upon prototypes of the genera associated with ics 2, 3, 4, and 6 are M-invariant. Again, in Eriksson's graphic representation, the centers of these genera lie directly on the axis of symmetry in question. (Oddly, Eriksson does not discuss the circle-of-fifths transform at all, despite the fact that many of his figures display this vivid symmetry.)

We have been emphasizing for some time that M_5 structures intergeneric affinities between $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, but have just noted that the prototype of the **other** four genera are all invariant under M_5. Recall now from our initial discussion of intergeneric affinities that the similarity of two chords p and q is exactly the same as the similarity of M_5p and M_5q. Suppose that p is invariant under M_5; then it follows that q and M_5q are each identically similar to p — which leads us immediately to the conclusion that that M_5 preserves the *intra*generic affinities of these other four genera. Thus Figure 1.11 may be taken as a demonstration not only of M_5's role in structuring the intergeneric affinities between $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, but also of the invariance of the intrageneric affinities of $\mathbb{Q}(12, 6)$ under M_5.

The algebraic genera of *SG*(3) — or, more particularly, the taxonomic relationships between *SG*(2) and *SG*(3) — engage all of the issues raised by qualitative genera, but from an orthogonal point of view. That being the case, we should be particularly interested in Morris's major question: What would *SG*(4) be? Each of the numbered set-group systems relates to its taxonomically inferior rank by adding to the canon of equivalence operators. Therefore such an *SG*(4) could be created by finding some other permutation of pcs (in addition to transposition, inversion, and *M*) and elevating it to the status of an equivalence operator. In particular, Morris is interested in operators that

> preserve certain interval-classes while changing others in the pcsets they transform. In this way the vectors of the mapped sets will be able to retain some of their features. As a result, the set-groups within the set-groups systems we will develop will be related by a kind of similarity measure. Fortunately, all of these strictures are inter-related; by actualizing one property we will gain the others (1982, pp. 113–13).

Such **nonstandard operators** will only be useful if they do not simply collapse all chords of a single cardinality into one category — Harrison's N operator does this, since, as discussed above, it can be used in combination with transposition and inversion to turn any chord into any other chord of the same cardinality.

Each of Morris's nonstandard operators begins with a partition of the aggregate into identical dyads, then maps each pc in each dyad into the other. For his α, for instance, he partitions the aggregate into six instances of ic 1. There is only one way to do this (up to transposition):

$$\{0,1\}\{2,3\}\{4,5\}\{6,7\}\{8,9\}\{A,B\}.$$

Note that in this partition, the ic-1 dyads are arranged along an "outer" ic-2 cycle. The α operator simply swaps the pcs within each of these dyads; another way to describe this is that each pc in the even whole-tone collection gets transposed up a semitone, and each pc in the odd whole-tone collection down a semitone. Immediately we can see that every prototype species of $\mathbb{Q}(12, 6)$ is necessarily invariant under α; it happens that the prototypes of $\mathbb{Q}(12, 3)$ are also invariant. One would do well to wonder whether these invariances generalize from the particular prototypes of these genera to their intrageneric affinities as a whole, as we saw to be the case with M_5's preservation of the intrageneric affinities of four qualitative genera.

The α operator does not leave any other generic prototypes invariant, but it does something suggestive to the prototypes of $\mathbb{Q}(12, 2)$ and $\mathbb{Q}(12, 4)$ — associated, respectively, with ics 6 and 3. Consider the tetrachordal prototypes, which are 4–9 [0167] and 4–28 [0369]. Thanks to transpositional and inversional invariances, there are only six exemplars of the former, and just three of the latter. All three of the 4–28 chords become 4–9 chords when transformed by α. Since α (like all of Morris's nonstandard operators) is its own inverse, those 4–9 chords — {1278}, {349A}, and {56B0} — transform into exemplars of 4–28 under α as well. The other three 4–9 chords — {0167}, {2389}, and {45AB} — are invariant under α. As a result, this pair of species constitutes a single algebraic genus under $SG(\alpha)$. Indeed, the prototypes of these two genera invariably relate to one another via α, with one exception: the hexachordal $\mathbb{Q}(12, 2)$ prototype, 6–7

OP.	PARTITION	GENERA WITH PROTOTYPES...	
		INVARIANT	SWAPPED
α	{0,1} {2,3} {4,5} {6,7} {8,9} {A,B}	$\mathbb{Q}(12,3)$ $\mathbb{Q}(12,6)$	$\mathbb{Q}(12,2)$, $\mathbb{Q}(12,4)$
β	{0,2}, {1,3}, {4,6}, {5,7}, {8,A}, {9,B}	$\mathbb{Q}(12,3)$	
γ	{0,3}, {1,4}, {2,5}, {6,9}, {7,A}, {8,B} {0,3}, {2,5}, {4,7}, {6,9}, {8,B}, {A,1}	$\mathbb{Q}(12,2)$ $\mathbb{Q}(12,4)$	
δ_1	{0} {1} {2} {3} {4,8} {5,9} {6,A} {7,B}	$\mathbb{Q}(12,3)$	$\mathbb{Q}(12,1)$, $\mathbb{Q}(12,4)$, $\mathbb{Q}(12,5)$
δ_2	{0} {1} {2} {7} {B,3} {4,8} {5,9} {6,A}	$\mathbb{Q}(12,3)$	
δ_5	{0} {1} {3} {6} {A,2} {4,8} {5,9} {7,B}	$\mathbb{Q}(12,3)$	$\mathbb{Q}(12,1)$, $\mathbb{Q}(12,4)$, $\mathbb{Q}(12,5)$

FIGURE 1.12. *Morris's nonstandard operators.*

{012678}, is transformed by α not into Hanson's hexachordal ic-3 projection, but into the closely related Petrushka chord, 6–30 [013679]. (But remember that our elevation of Hanson's projections to prototype status is not infallible, and that 6–30 is a maxpoint of ic 6, among other things.) The obvious question concerning affinities is whether α describes intergeneric affinities between $\mathbb{Q}(12,2)$ and $\mathbb{Q}(12,4)$.

Without delving deeply into Morris's theory, we observe that the other operators are constructed likewise (beginning with other homogeneous partitions of the aggregate), and produce similarly interesting results. The table in Figure 1.12 lists the six nonstandard operators Morris uses to construct set-group systems. As α is based on a partition into ic-1 dyads, β and γ stem from ic-2 and ic-3 partitions, respectively. The aggregate cannot be partitioned into six discrete instances of ic 4, so Morris constructs a family of δ operators based on best-case partitions, of which only three generate unique set-group systems.

It should be observed that the intervals (and corresponding prototypes) preserved under each operator are not necessarily the same as those involved in the partition underlying each operator. We saw that the ic-1 partition underlying α is best understood as being structured by an ic-2 cycle; thus it is the ic-2 and ic-4 prototypes that are preserved. Similarly, the ic-2 partition underlying β is structured by a hexatonic scale, a prototype of the genus $\mathbb{Q}(12,3)$, all of whose prototypes are β-invariant. The two possible ic-3 partitions (generating γ) are structured by 6–7 [012678], a prototype of $\mathbb{Q}(12,2)$, and the whole-tone collection, with appropriate invariances ensuing.

Each one of the six nonstandard operators leaves some array of generic prototypes invariant, raising the possibility of intrageneric-affinity preservation. Three (α, δ_1, and δ_5) consistently conflate prototypes of different genera into the same set-groups, raising the possibility of intergeneric affinities. (Since each operator is a "root" of M_5, each conflates prototypes of $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, which is not mentioned in the table.) Insofar as we have had some general success in describing affinities in terms of prototypes and similarity relations, the questions just raised about the relationship between Morris's set-group systems and the affinities of the qualitative genera are empirical ones. The event is that the nonstandard operators do not, on the whole, work as neatly as we might have hoped in the general case, despite their tidy treatment of the prototypes.

1.3.3 Cohn's cyclic approach. As an adjunct to his theory of transpositional invariance, Cohn (1991) develops five set-group systems of his own. Each is based on a special kind of nonstandard multiplicative operator he calls a *CYCLE homomorphism*, q.v. also Starr (1978). The "M_2" operator we discussed above in connection with Harrison's N operator is one of these (CYCLE-6). Cohn is interested in taking advantage of precisely the same characteristic of these operators that Harrison viewed as a problem — specifically, that they do not necessarily preserve cardinality. A simple example, which we have already considered, concerns the transformation of a tritone under CYCLE-6 ("M_2"), which becomes a single pc. From a certain point of view, there is something missing from this statement — one might like to think that the CYCLE-6 transform of a tritone is actually two "copies" of the same pc — but in the usual framework of pcset theory, which treats chords as classical sets of pitch-classes, there is no technical facility for making counting "copies" of a pc in a chord. Lewin (1977) and Morris (1998) have both suggested that such a facility, sometimes known as a *multiset*, might come in handy in certain theoretical contexts, but the important and fascinating issues involved constitute a can of worms best left closed in the present context.

Figure 1.13 lists the five CYCLE homomorphisms, along with the multiplicative operators they closely resemble. In Cohn's conception, the homomorphisms "act not

	0	1	2	3	4	5	6	7	8	9	A	B	
"M_2"	0	2	4	6	8	A	0	2	4	6	8	A	
CYCLE-6	0	1	2	3	4	5	0	1	2	3	4	5	(mod 6)
"M_3"	0	3	6	9	0	3	6	9	0	3	6	9	
CYCLE-4	0	1	2	3	0	1	2	3	0	1	2	3	(mod 4)
"M_4"	0	4	8	0	4	8	0	4	8	0	4	8	
CYCLE-3	0	1	2	0	1	2	0	1	2	0	1	2	(mod 3)
"M_6"	0	6	0	6	0	6	0	6	0	6	0	6	
CYCLE-2	0	1	0	1	0	1	0	1	0	1	0	1	(mod 2)
"M_{12}"	0	0	0	0	0	0	0	0	0	0	0	0	
CYCLE-{1,5}	0	0	0	0	0	0	0	0	0	0	0	0	(mod 1)

FIGURE 1.13. *Cohn's CYCLE homomorphisms as multiplicative operators.*

only upon individual pitch classes, but on all mod-12 pc sets as well, mapping them into set classes within smaller universes." These smaller universes are represented by the integers modulo *n*, where *n* is the number of the CYCLE homomorphism. There is an important, but subtle, shift of perspective contained in Cohn's invocation of smaller universes here; it is most clearly manifest in the mismatch between the numerical labels of the "M_m" operators and their associated CYCLE-*n* homomorphisms.

Cohn captures this shift of perspective in a diagram adapted here as Figure 1.14. The lighter circles in the diagram represent interval cycles; they also represent the collections of pcs that map identically under the relevant CYCLE homomorphism. The heavier circles represent what Cohn calls "cycles of cycles" (p. 12). This terminology refers to the fact that the CYCLE homomorphisms involve collapsing interval cycles into a representative indicated here with large type; this representative "may be thought of as not only furnishing a label for that cycle, but as vortically inhaling its contents" (p. 14) under the relevant homomorphism. Once the interval cycles are collapsed, it is clear that the cycles are themselves cyclically related to each other; this accounts for the periodic nature of the mappings detailed in Figure 1.13, clarified there with vertical lines. The "cycles of cycles" are Cohn's "smaller universes."

In simple terms, this shift of perspective calls into question the usual interpretation of multiplicative operators and subjects them to a sort of figure-ground reversal

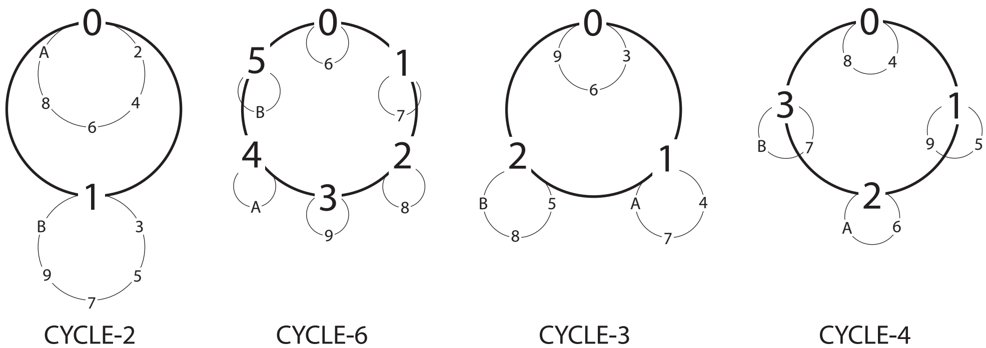


FIGURE 1.14. *Adapted from Cohn (1991), Table 5.*

— conceptualizing them not as things that expand intervals within the fixed frame of octave equivalence, but as things that preserve intervals while shrinking the universe. Each of the CYCLE homomorphisms, as Cohn uses them, extends the idea of octave equivalence to smaller intervals. CYCLE-2 envisions a world of whole-tone equivalence; CYCLE-3, minor-third equivalence; CYCLE-4, major-third equivalence; and CYCLE-6, tritone equivalence. Cohn's set-groups are nothing more than chord species of the usual kind — equivalence classes of classical sets of pitch-classes under transposition and inversion — but under these novel notions of pitch-class equivalence.

Cohn gets to this point, as we have mentioned, through a study of transpositionally invariant chords. Many of these are Hanson projections — all, in fact, besides the French-sixth chord 4–25 [0268] and the Petrushka chord 6–30 [013679] — and are therefore depicted in Figure 1.2, where the interior of their clockface diagrams are colored gray. Inspection of these graphical representations highlight Cohn's point: a transpositionally invariant set exhibits a repeated pattern that is periodic at one of the intervals of equivalence mentioned above. The hexatonic collection 6–20 [014589], for instance, a prototype of what we are calling $\mathbb{Q}(12, 3)$, is periodic at the major third. Since the pattern is periodic modulo 12, its structure can be completely, and more efficiently, described in terms of a nonperiodic pattern in a smaller universe; the mapping from the twelve-pc universe down to the smaller universe is provided by a CYCLE homomorphism.

We are now in a position to discuss the mismatch between our labels for the qualitative genera and the labels for the interval-classes associated with them. This is precisely the same as the mismatch noted above between the numerical labels of the "M_m" operators and their associated CYCLE-*n* homomorphisms. The hexatonic collection is periodic at the major third. This interval being a representative of ic 4, the CYCLE-4 homomorphism provides the mapping into the requisite smaller universe (major-third equivalence). When we characterize CYCLE-4 as "M_3," or when we assert the species 6–20 as a prototype of the $\mathbb{Q}(12, 3)$ genus, we are making reference to the fact that there are three major thirds to the octave. Similar statements could be made about, for example, CYCLE-2, ic-2 equivalence, and the transpositionally invariant whole-tone collection 6–35 [02468A], on the one hand, and "M_6," six whole-tones to the octave, and 6–35 as a prototype of $\mathbb{Q}(12, 6)$ on the other. The clockface diagrams in Figure 1.2 are each supplemented with markings that show their division into *q* equal parts, where *q* is the index of the qualitative genus $\mathbb{Q}(12, n)$.

Cohn's approach opens up a number of interesting avenues of inquiry in relation to the issues that have been placed on the table so far. The various approaches we have considered all seem to suggest, in one way or the other, that intergeneric affinities (the structural relationships between qualitative genera) have something to do with multiplication and interval cycles. Moreover, the prototypes laid out in Figure 1.2 — which, despite the fact that Hanson conceived them intervallically, have strongly resonanted throughout our survey of different approaches to chord quality — seem to warrant further inquiry from the point of view of transpositional invariance, given how many of them have that symmetry. Cohn's work appears to tie all of these ideas together, suggesting that chord quality has something to do with equal divisions of the octave, but it raises two important questions of its own.

First, there are five CYCLE homomorphisms and six qualitative genera. CYCLE-{1,5}, which simply collapses all pcs into one category under semitone- or fifth-equivalence, corresponds in some vague sense to $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, if only conceptually, but these are precisely the genera whose only transpositionally invariant prototypes are trivial — the empty chord and the aggregate — and also in violation of UGP, if trivially so. What the prototypes of these genera have in common is something we have not yet treated adequately: each is a contiguous segment of the chromatic scale in the case of $\mathbb{Q}(12, 1)$, or the circle of fifths in the case of $\mathbb{Q}(12, 5)$.

Second, the concept of a one-to-one mapping between interval classes and divisions of the octave, which may be implied by the strong connection between Cohn's approach and our working set of prototypes derived from Hanson's intervallic approach, is tenuous at best. Our best example of this in the twelve-pc universe has to do with $\mathbb{Q}(12, 5)$ — the octave is not equally divisible into 5 equal parts. Nonetheless, it is well-known that certain of our prototypes for $\mathbb{Q}(12, 5)$, viz. the pentatonic collection 5–35 [02479] and its complement, the diatonic collection), are *maximally even*; they represent the *best* way to distribute five pcs (or five absences-of-pcs) equally around pitch-class space. As it happens, these chords can also be generated by ic 5 — what may come as a disappointing surprise, as we will see in the next chapter, is that this property does not generalize in the obvious way (see Carey and Clampitt, 1996).

Having completed our survey of existing approaches to qualitative taxonomy, we will change course now, pursuing the promising idea of equal divisions of a pitch-class universe, which is covered by the theory of maximally even chord species. This will be the key to fitting together some of the seemingly incompatible approaches we have covered here.

CHAPTER 2

A unified theory of generic prototypes

Now that we have found common cause among all of the various approaches to chord quality we have surveyed, and seen that they are rather tightly interrelated to one another, the complexity of the theoretical task ahead should be clear. It seems that three abstract concepts — prototypes, intrageneric affinities, and intergeneric affinities — can provide the infrastructure of a theory of chord quality; our task is to make these concepts concrete in the domain of chords and species.

Rather than starting from the bottom up, proceeding from any one of the approaches covered in Chapter 1, we will take a top-down approach, wiping the slate clean and taking as our point of departure a theoretical framework from scale theory (Clough and Douthett, 1991), the original intent of which was to generalize certain aspects of the diatonic scale. (In the broadest sense, that is our intent as well: the diatonic scale, after all, is a generic prototype.) As we generalize Clough and Douthett's work outward into a theory of generic prototypes, we will find ourselves subsuming certain key notions from each of the approaches that have been covered so far, fitting them into the emerging unified framework. At times the work will, of necessity, be highly technical. The business of Chapter 3 will be to develop an approach to affinities that interfaces with this theory of prototypes; our treatment of that approach will retroactively clarify and simplify much of the technical language that comes up below.

§ 2.1 Maximally even subgenera.

Although it is assumed that the reader is familiar with Clough and Douthett's remarkable paper on maximally even chords and chord species, we will begin by summarize and recasting some of their results, periodically adding some new terms concepts into the mix. An asterisk (*) denotes a term or usage original to this work. Page and theorem numbers in this section refer to Clough and Douthett's paper unless otherwise specified. Following Clough and Douthett's usage, the variable c will stand for the number of pcs in a given pitch-class universe; the variable d will generally stand for the cardinality of a chord or species.

A **maximally even (ME)** chord is one "whose elements are distributed as evenly as possible around the chromatic circle" (p. 96). There is at least one ME chord of every cardinality in every chromatic universe (Theorem 1.2, p. 102). Any transposition of a ME chord is also ME, and conversely, all ME chords of a given cardinality in a given chromatic universe are related by transposition (Theorems 1.6 and 1.7, p. 108). Furthermore, all ME chords are inversionally invariant (Theorem 1.8, p. 109).

The foregoing may be summarized by observing that all ME chords of the same cardinality in the same chromatic universe belong to the same inversionally invariant species, and, conversely, that all members of that species are ME. Let $M(c, d)$ stand for the unique ME species of cardinality d in a universe of c pcs. A ME chord is a **singleton** if $d = 1$, an **antisingleton*** if $d = c - 1$, and **trivial** if $d = 0$ or $d = c$. Singletons and antisingletons (and various equivalence classes of them) are **semitrivial*** (but not trivial).

We recall that the abstract complement of a ME species is also ME (Theorem 3.3, p. 150; Corollary 3.2, p. 151). The chords in a ME species, together with their complements, constitute a **ME subgenus***. Let $\mathbb{M}(c, d)$ stand for the ME subgenus including $M(c, d)$ and its complement.

The number of ME subgenera in a c-pc universe is equal to $c/2$ rounded down to the nearest integer. (This is the same as the number of interval classes in the universe.) As a preview of what is to come, our theory of chord quality will entail making a one-to-one correspondence between nontrivial ME subgenera and qualitative genera; a sense of this can be had by inspecting Figure 1.2 (in which ME species are indicated with thicker circles) and considering the relationship between the species of $\mathbb{M}(c, d)$ and the other prototypes of $\mathbb{Q}(c, d)$.

A stern warning is in order at this point: although in any pitch-class universe, the number of ME subgenera is equal to the number of interval classes, there is no natural and general way to draw a one-to-one correspondence between them. We issue the warning because it *happens to be possible* to do so in the usual 12-pc universe, which leads to the fallacious notion that the interval content of a chord is a cause (and not a symptom) of its qualitative relationship to other chords. From time to time we will refer to this notion as the **Intervallic Half-Truth**. To understand the Intervallic Half-Truth in sufficient depth, we need to pursue the question of how interval classes become associated with ME chords in the first place. This will require a small amount of new work relating to the classification of ME species.

It is established above that $M(c, d)$ exists for any c and d, and that its exemplars are inversionally invariant. Clough and Douthett show that, $M(c, d)$ will have at least one of the following two additional properties, depending on the greatest common factor of c and d:

(TINV) If and only if $\mathrm{gcf}(c, d) > 1$, exemplars of $M(c, d)$ are **nontrivially transpositionally invariant**, and the **index***, or number of distinct exemplars, of $M(c, d)$ is given by $c/\mathrm{gcf}(c, d)$. (Theorem 1.9, p. 113)

(G) If and only if $\mathrm{gcf}(c, d) = 1$ or $\mathrm{gcf}(c, d) = d$, $M(c, d)$ is **generated** (Theorem 3.1, p. 148). That is, for any exemplar, there is at least one way of ordering the constituent pcs (up to reversal) such that any two adjacent pcs are separated by the same interval — as, for instance, one can order a diatonic collection as a contiguous segment of the circle of fourths/fifths. Such an ordering will be called a **generated ordering**; the interval between adjacent pcs of a generated ordering is the **generating interval** and the interval between the last and first pcs of a generated ordering is the **wraparound interval**.

We will use these properties, in conjunction with the theory of complementation, as the basis of a fourfold classification of ME species that is related to Clough and Douthett's threefold classification (p. 148), but slightly stricter and more germane to our overarching theoretical goals.

In a later article, Clough et al. (1999, Theorem 3.2, pp. 91–93) prove that all ME species that are G are also well-formed (WF) in the sense of Carey and Clampitt (1989), meaning that each instance of the generating interval spans the same number of scale steps — e.g., the generating fifths of a diatonic collection span five scale steps, which is why they are called *fifths*. We mention this in order to take advantage of the latter's distinction between what they call *degenerate* and *nondegenerate* WF species, which distinction features prominently in the former's proof. All WF species (ME or not) are G. **Degenerate WF** species have the property that in any generated ordering, the generating interval is equal to the wraparound interval, guaranteeing that they will be TINV; **nondegenerate WF** species have the property that in any generated ordering, the generating interval is not equal to the wraparound interval, guaranteeing that they will not be TINV. There is no third option; every WF species is either degenerate or nondegenerate. Here is how the distinction carries over to ME species:

Class I If and only if c and d are coprime (i.e., $\text{gcf}(c, d) = 1$), then $M(c, d)$ is G and nondegenerate WF. Because c is not evenly divisible by d, it is ME only in the sense that its constituent pcs are "distributed *as evenly as possible* around the chromatic circle." The fact that every possible generating interval is different from the corresponding wraparound interval precludes the possibility that $M(c, d)$ is TINV. In the event that $M(c, d)$ is semitrivial (that it is a singleton or anti-singleton) it is certain that $\text{gcf}(c, d) = 1$, and that the nondegenerate status claimed for $M(c, d)$ is, since the existence of generating and wraparound intervals is somewhate questionably, strictly by fiat.

Class IIa If and only if d divides c (i.e., $\text{gcf}(c, d) = d$) and $d > 1$, then $M(c, d)$ is G and degenerate WF. Because d divides c evenly, its constituent pcs are distributed perfectly evenly around pitch-class space, in contrast to a Class I species. $M(c, d)$ is therefore TINV.

In the usual pitch-class universe, the ME species of Class I (G and TINV) are either singletons or antisingletons, or pentatonic collections or their complements (diatonic collections). The ME species of Class IIa (G and not TINV) are tritones, augmented triads, diminished-seventh chords, or whole-tone collections.

Having exhausted the cases in which ME species are G, our classification of ME species will be completed by cases that are TINV but not G. These we will divide into two types distinguished primarily by their behavior under complementation. We have already mentioned that the complement of any ME chord is itself ME; consequently any species $M(c, d)$ is the abstract complement of $M(c, c - d)$. An elementary theorem of number theory holds that $\text{gcf}(c, d) = \text{gcf}(c, c - d)$. It follows that the (abstract) complement of a Class I species is the complement of another Class I species, since in that case $\text{gcf}(c, d) = \text{gcf}(c, c - d) = 1$.

Suppose $M(c, d)$ is not G; then $1 < \text{gcf}(c, d) < d$ (Theorem 3.1, p. 148) and $M(c, d)$ is TINV (Theorem 1.9, p. 113–14). All such species can be divided into two types:

Class IIb If and only if $1 < \text{gcf}(c, d) < d$ and $\text{gcf}(c, d) = c - d$, then $M(c, d)$ is the complement of a Class IIa species, and we say that it is **complement-generated* (CG)**, which means simply that its complement is generated. Note that the complement of a Class IIa species is of Class IIb if and only if d is not exactly half of c, as is the case with the usual whole-tone collections (Theorem 3.4, p. 151–52). In such cases the complement of a Class IIa species is "another Class IIa species" in a somewhat trivial sense. Diverging from Clough and Douthett's practice, we will allow for purposes of argument that all CG species have generating intervals — the same as that of their generated complements — even though they are not generated, strictly

CLASS	DEFINITION	G?	CG?	TINV?	COMPL.
I	$\text{gcf}(c, d) = 1$	yes	yes	no	I
IIa	$\text{gcf}(c, d) = d \neq 1$	yes	no	yes	IIb*
IIb	$\text{gcf}(c, d) = c - d \neq 1$	no	yes	yes	IIa
III	otherwise	no	no	yes	III

*Unless $c = d/2$, in which case $M_{c,d}$ is its own complement (up to transposition).

FIGURE 2.1. *Classification of all ME species for $c > 1$ and $0 < d < c$.*

speaking, unless d is exactly half of c (for certain Class IIa species) or c and d are coprime (Class I).

Class III If and only if $1 < \text{gcf}(c, d) < d$ and $\text{gcf}(c, d) \neq c - d$, then $M(c, d)$ is not CG. Class III species are therefore the complements of other Class III species. They are found in most pitch-class universes where c is composite (not prime) and sufficiently large, but not in the usual 12-pc universe.

Classes IIb and III complete the classification of ME species, which is summarized in Figure 2.1. Trivial ME species — null pcsets and their complements — are excluded from the classification (and from further discussion), although the compulsive reader is welcome to classify them as Class I at the cost of slightly messier definitions throughout the system. This classification system is closely related to Clough and Douthett's; the major difference is that their system conflates our Classes IIb and III.

Thanks to the observations made about complementation, the classification system extends readily to ME subgenera. The complement of a Class I (respectively, Class III) ME chord is a Class I (Class III) ME chord; the two belong to a Class I (Class III) ME subgenus. The complement of a Class IIa ME chord belongs to either Class IIa or Class IIb; the complement of a Class IIb ME chord always belongs to Class IIa. For this reason we refer to the relevant ME subgenera simply as Class II, and largely drop the distinction between Class IIa and Class IIb for ME species.

§ 2.2 Against the Intervallic Half-Truth.

In order to add flesh to the bones of the foregoing discussion, we will now study some specific examples of ME species. Due to the generality of the theory (evidenced by the fact that there are no Class III ME species in the usual twelve-pc universe) we will find ourselves exploring some other small pitch-class universes in order to illustrate some of the relevant issues. Our argument against the Intervallic Half-Truth necessitates this move, since certain aspects of Class III ME species provide the best point of entry into a deep understanding of the problems with the pernicious belief that underlies the most common theoretical approach to chord quality.

2.2.1 Examples. Figure 2.2 lists the ten nontrivial ME species in the 11-pc universe ($c = 11$), illustrating each with a modified-clockface display of a representative. Because c is prime, it has no common factors with any possible d, and gcf(c, d) is always equal to 1. Consequently, every ME species in this universe is necessarily of Class I — it is G, nondegenerate WF, and non-TINV. The rightmost column of the table lists the generators of each species. Generators come in pairs, each of which constitutes an interval class.

The clockface diagrams can be used to vividly demonstrate the complement relations that hold between $M(c, d)$ and $M(c, c - d)$ by reconceptualizing the figure-ground relationship from black-on-white to white-on-black. This way of thinking is useful in conceptualizing the role of a generator in a large ME species. In $M(11, 8)$, for example, the fact that the individual pcs of an exemplar can be generated by a 4- or 7-cycle is in some sense less relevant than the fact that (shifting perspective) the *missing* pcs can also be so generated — which means that we can understand the species itself (shifting perspective again) as being composed of three *clusters* arranged along a 4- or 7-cycle. This line of thought encourages an interpretation of $M(11, 8)$ not as a complement of $M(11, 3)$, but as a “smudged” or “blurred” version of it (in a somewhat Hansonian sense), a characteristic of the relationship between the complementary members of a

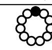
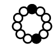
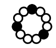
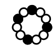
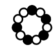
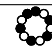
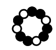
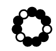
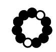
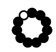
d	gcf(c, d)	c/gcf(c, d)	d/gcf(c, d)	$M_{c,d}$	CLASS	GEN(S).
1	1	11	1		I	(all)
2	1	11	2		I	5/6
3	1	11	3		I	4/7
4	1	11	4		I	3/8
5	1	11	5		I	2/9
6	1	11	6		I	2/9
7	1	11	7		I	3/8
8	1	11	8		I	4/7
9	1	11	9		I	5/6
10	1	11	10		I	(all)

FIGURE 2.2. *The species $M(c, d)$ for $c = 11$.*

ME subgenus. In Figure 2.2, a horizontal line separates the two instances of each ME subgenus, which are symmetrically arrayed above and below the line.

Note particularly the following half-truth: each ME subgenus can be associated with one interval-class generator, and that each interval class is the generator of a unique ME subgenus. This is a half-truth because it ignores the fact that (technically) any interval at all is a generator of $M(c, 1)$ and any generator of some $M(c, d)$ that is Class I is also a generator of $M(c, c-1)$ (Theorem 3.2, p. 149). We can accept the half-truth if we ignore this technicality and assert that interval class 1/10 is the unique generator of $M(11, 1)$ and $M(11, 10)$. This would have a payoff in the form of a one-to-one correspondence between interval classes and ME subgenera; but this is the Intervallic Half-Truth.

Figure 2.3 illustrates the ME species in the familiar 12-pc universe ($c = 12$). The reader should take some time to consider the idea of complementation as smudging in this context of more familiar species — interpreting the octatonic collection as a smudged diminished-seventh chord, or the diatonic collection as a (very slightly) smudged pentatonic collection. This point of view provides a hint as to how the present line of inquiry will eventually connect back to Figure 1.2, since the generic prototypes

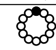
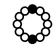
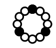
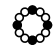
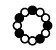
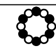
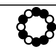
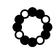
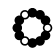
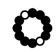
d	gcf(c, d)	c/gcf(c, d)	d/gcf(c, d)	$M_{c,d}$	CLASS	GEN(S).
1	1	12	1		I	(all)
2	2	6	1		II	6
3	3	4	1		II	4/8
4	4	3	1		II	3/9
5	1	12	5		I	5/7
6	6	2	1		II	2/10
7	1	12	7		I	5/7
8	4	3	2		II	3/9
9	3	4	3		II	4/8
10	2	6	5		II	6
11	1	12	11		I	1/11 (also 5/7)

FIGURE 2.3. *The species $M(c, d)$ for $c = 12$.*

that are not ME all appear to be (evenly) smudged versions of their characteristic ME subgenera. We will shortly discuss a more formal way of describing such smudges.

The number 12 is not prime, and so we find some ME species of Class II in the universe. These, in contrast to Class I species, are TINV, and they follow the rules of complementation laid out above — each Class IIb species is complement-generated, since it is the complement of a Class IIa species, all of which are generated. (The usefulness of the CG property should be clearer now that we have spoken about complementation as smudging.) Furthermore, every Class IIa species is the complement of a Class IIb species with the exception of $M(12, 6)$, the whole-tone collection, which (since d is precisely half of c) is a transposed form of its own complement.

Once again, it is possible to make a one-to-one association between interval classes and ME subgenera by virtue of the fact that all ME species in this universe are either G or CG — provided one succumbs to the Intervallic Half-Truth and accepts the half-truth about singletons and their complements being generated only by the intervals 1 and $c - 1$.

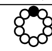
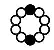
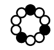
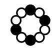
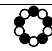
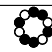
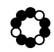
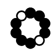
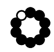
d	gcf(c, d)	c/gcf(c, d)	d/gcf(c, d)	$M_{c,d}$	CLASS	GEN(S).
1	1	10	1		I	(all)
2	2	5	1		II	5
3	1	10	3		I	3/7
4	2	5	2		III	—
5	5	2	1		II	2/8 (also 4/6)
6	2	5	3		III	—
7	1	10	7		I	3/7
8	2	5	4		II	5
9	1	10	9		I	1/9 (also 3/7)

FIGURE 2.4. *The species* $M(c, d)$ *for* $c = 10$.

Consider now Figure 2.4, which displays ME species in the 10-pc universe and provides us with our first encounter with ME species of Class III. These are the particular kind of ME species that are neither G nor CG, and so they cannot be associated with generating intervals, even in the looser sense that we have been doing this with CG species. As a result, it is not even possible to utter the half-truth about a one-to-one correspondence between interval classes and ME subgenera in this universe. Even given a faulty association of ic 1/9 with $M(10, 1)$ and $M(10, 9)$ we cannot associate ic 4/6 uniquely with any of the ME subgenera. The process of elimination would suggest the ME subgenus associated with the two Class III species ($M(10, 4)$ and $M(10, 6)$), but it is difficult to ignore the fact that $M(10, 4)$ contains no instances of ic 4/6 whatsoever, and that the two instances contained in $M(10, 6)$ are attributable to the Generalized Hexachord Theorem.

Having finally come up with a serious counterexample to the Intervallic Half-Truth, let us study it carefully. Note first of all that the interval class in question (4/6) is not truly available to generate these Class III species, as it as already occupied with $M(10, 5)$ — as Figure 2.4 shows, interval class 2/8 is not the only possible generator of $M(10, 5)$. In the course of proving their only theorem about the generation of Class IIa species (Theorem 3.1, p. 148), Clough and Douthett only prove that the generators of a Class

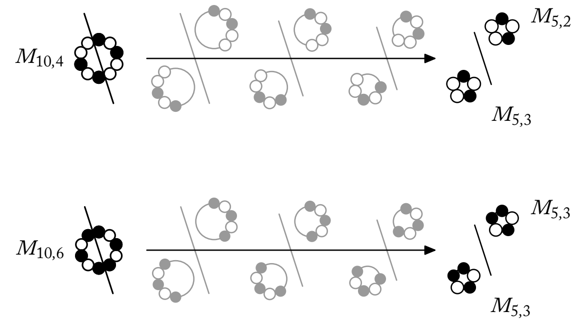


FIGURE 2.5. *Class III ME chords can be split into gcf(c, d) repeated "copies" of some nontrivial Class I chord.*

IIa $M(c, d)$ include c/d and $c - c/d$ (in this case, 2 and 8, respectively, as shown in Figure 2.4); they do not offer a technique for enumerating all possible generators.

Furthermore, recall that all Class III species are TINV. All ME species that are TINV are invariant under all and only transpositions by multiples of $c/\text{gcf}(c, d)$ "semitones" (Theorem 1.9, p. 113–14). In the instant case, $M(10, 4)$ and $M(10, 6)$ are invariant when transposed by $5 = c/2$ "semitones." This is thanks to the fact, illustrated in Figure 2.5, that each can be split into two identical half-octave parts, each of which is an instance of a generated ME subgenus with $c = 5$ and a generating interval class of 2/3. There is therefore a sense in which the Class III species in question are generated after all — but by a two-stage process; first the octave is cut into two equal pieces with a generating interval of 5, and then each of the pieces is treated as a universe unto itself in which an ME set is generated with a generating interval of 2 or 3. Or, looking at it from the opposite point of view, we begin with a Class I species in the 5-pc universe, and then project it into the 10-pc universe using transpositional combination (TC), a topic we will take up shortly.

We have seen that Class III species are not generated strictly speaking, nor are they generated in the extended sense that ME species with the CG property (Class IIb) are generated. (They are, however, "doubly generated" from the point of view espoused in the previous paragraph, as are Class IIb species.) It is for just this reason that they constitute the crux of the Intervallic Half-Truth. This leaves us the rather pressing question of how we can characterize ME subgenera (and, by extension, the

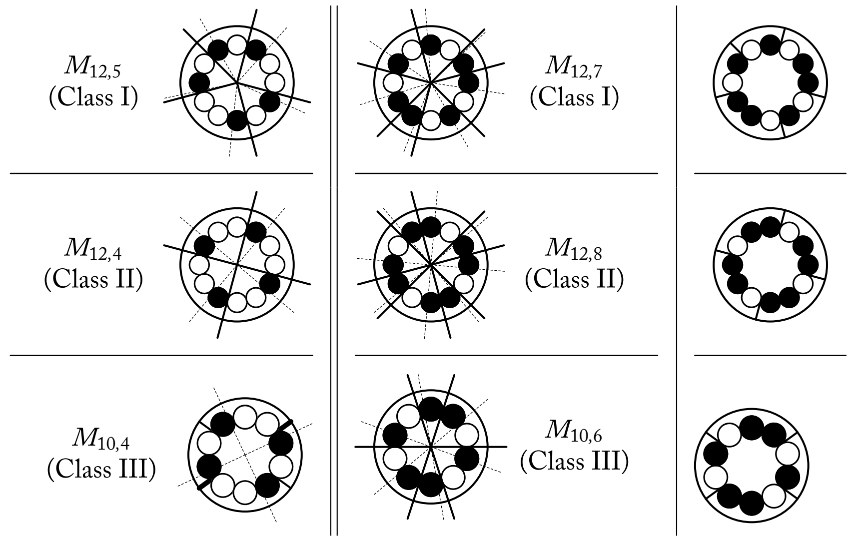


FIGURE 2.6. *ME subgenera and aligned subuniverses.*

very qualitative genera we are trying to construct) without reference to the notion of a generating interval.

The answer is suggested by Figure 2.6. Clough and Douthett establish early on that all and only ME chords, which they initially define intervallically, admit description in terms of the function $J^m_{c,d}(N) = \lfloor \frac{cN+m}{d} \rfloor$, as N varies from 0 to $d-1$ (Theorem 1.2, p. 102; Theorem 1.5, p. 108). Essentially, this means that the constituents of an ME chord are determined by dividing pitch-class space into d equal parts, then rounding down to the nearest pc, with the unbound variable m simply determining where the first "slice" is made. Examples of this determination are shown in the first two columns of Figure 2.6, which include examples of each of the four classes of ME chords, together with literal examples of the complementation relations we have discussed above. In the figure, which uses the style of clockface diagrams used throughout this work, the rounding process is imagined in terms of each pitch-class beginning at the counterclockwise edge of the appropriate black or white circle and extending to the point just before its clockwise edge, which edge coincides with the the beginning of the next-higher pitch-class. Thin dashed lines show the equal division of pitch-class space; thicker solid lines show the result of the rounding process, which (in all and only cases where $M(c, d)$ is neither Class IIa nor trivial) results in an uneven — but maximally even — division.

The rightmost column of Figure 2.6 superimposes the maximally even division of the smaller chord of each pair on the diagram of its (larger) complement in order to make clear the following crucial principle that holds for any $M(c, d)$: regardless of its cardinality d, let d' be the lesser of $c - d$ and d itself; there exists a maximally even way to divide pitch-class space into d' subspaces such that each subspace contains a trivial ME chord. Furthermore, these subspaces are "aligned" with respect to one another insofar as (to put it in the visual language of our figure) the white pcs are always on the counterclockwise side of the subspaces if $d = d'$ and on the clockwise side if $d \neq d'$.

All of this is a rather more substantial way of saying something observed earlier in colloquial terms: that ME species that are larger than their complements can be viewed as "smudged" versions of their complements. We have now refined this by means of two important ideas. First, we know what exactly is being smudged: a maximally even division of pitch-class space into subspaces, each of which contains a singleton that is aligned with the singletons in the other subspaces. Second, we know exactly how the smudging is taking place: the singletons, which are one type of trivial ME chord, are transformed into antisingletons, which are the other type of trivial ME chord.

2.2.2 Argument. When last we left our theory of qualitative genera, it amounted only to the assertion that in any pitch-class universe, there are as many qualitative genera as there are nontrivial ME subgenera. It is time now to bring the theory up to date with our classification and characterization of ME chords. Each qualitative genus $\mathbb{Q}(c, q)$ (where $0 < q \leq c/2$) counts among its prototypes the species of its **characteristic ME subgenus**, $\mathbb{M}(c, q)$, as well as (more or less by default) the singletons and antisingletons of the semitrivial ME subgenus $\mathbb{M}(c, 1)$, which is also the characteristic ME subgenus when $q = 1$.

A qualitative genus may be classified as Class I, II, or III by virtue of its characteristic ME subgenus. Thus, for $0 < q \leq c/2$, the genus $\mathbb{Q}(c, q)$ belongs to

Class I if and only if $\text{gcf}(c, q) = 1$;

Class II if and only if $\text{gcf}(c, q) = q$; and

Class III otherwise.

We are working toward a way to establish prototypes of each cardinality for each qualitative genus, a project that will be completed in the next section. Thus far, we have been assuming that Hanson's projection procedure does this job; unfortunately, it only works reliably in universes (like the usual one) without Class III genera. The table in Figure 2.7 demonstrates two ways in which Hanson's technique fails in a Class III environment, using the now-familiar example of the ten-pc universe.

The first failure concerns the prototypes of $\mathbb{Q}(10,2)$ and $\mathbb{Q}(10,4)$, which take up the first two columns of the table. Under $\mathbb{Q}(10,2)$ we find the results of projecting ic 5 by Hanson's procedure, with the usual "shift" of 1 generating what we referred to in Figure 1.1 as the outer cycle. This seems to work fine; note in particular the structural resemblance of these prototypes to the Hanson prototypes of $\mathbb{Q}(12,2)$ in Figure 1.2 (the ic-6 projection in the usual 12-pc universe). The prototypes of $\mathbb{Q}(10,4)$ were also generated by projecting ic 5, but this time with a "shift" of 2. As it turns out, a redundant ic-5 projection is the only way to "doubly generate" $M(10,4)$ and its complement — $[0257]_{10}$ and $[013568]_{10}$, respectively — in the sense described above. Whereas in the usual 12-pc universe (and all other Class III–free universes) each nonterminating Hanson projection corresponds to only one qualitative genus — allowing us to set the "shift" uniformly at 1 — this is not generally the case.

The second failure is, in a sense, the reciprocal of the first, and concerns the prototypes of $\mathbb{Q}(10,5)$. This time, the characteristic ME subgenus, consisting of the self-complementary species $[02468]_{10}$, can be generated by two distinct interval-classes: 2 and 4. Under the columns headed "(ic 2)" and "(ic 4)" in Figure 2.7 we find the projections of those two ics; arrows indicate cardinalities at which they are identical (up to species). Which are the "right" prototypes of the qualitative genus? We have no choice but to answer that they both are, and to give up UPP (§1.2).

We see that $\mathbb{Q}(10,2)$ and $\mathbb{Q}(10,4)$ are both associated with ic 5, and that ics 2 and 4 are both associated with $\mathbb{Q}(10,5)$; here is a very strong counterexample to the Intervallic Half-Truth. The reciprocal relationship between the interval-class numbers

$\mathbb{Q}(c, d)$ $\mathrm{sig}(c, q)$ $\mathrm{sog}(c, q)$	$\mathbb{Q}(10, 2)$ 5 1	$\mathbb{Q}(10, 4)$ 5 2	$\mathbb{Q}(10, 5)$ 2 1	
	(ic 5)		(ic 2)	(ic 4)
0	$[]_{10}\Longrightarrow$	$\Longleftarrow[]_{10}\Longrightarrow$	$\Longleftarrow[]_{10}\Longrightarrow$	$\Longleftarrow[]_{10}$
1	$[0]_{10}\Longrightarrow$	$\Longleftarrow[0]_{10}\Longrightarrow$	$\Longleftarrow[0]_{10}\Longrightarrow$	$\Longleftarrow[0]_{10}$
2	$[05]_{10}\Longrightarrow$	$\Longleftarrow[05]_{10}$	$[02]_{10}$	$[04]_{10}$
3	$[015]_{10}$	$[025]_{10}$	$[024]_{10}$	$[026]_{10}$
4	$[0156]_{10}$	$[0257]_{10}$	$[0246]_{10}\Longrightarrow$	$\Longleftarrow[0246]_{10}$
5	$[01256]_{10}$	$[02357]_{10}$	$[02468]_{10}\Longrightarrow$	$\Longleftarrow[02468]_{10}$
6	$\overline{[0156]}_{10}$	$\overline{[0257]}_{10}$	$\overline{[0246]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[0246]}_{10}$
7	$\overline{[015]}_{10}$	$\overline{[025]}_{10}$	$\overline{[024]}_{10}$	$\overline{[026]}_{10}$
8	$\overline{[05]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[05]}_{10}$	$\overline{[02]}_{10}$	$\overline{[04]}_{10}$
9	$\overline{[0]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[0]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[0]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[0]}_{10}$
10	$\overline{[]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[]}_{10}\Longrightarrow$	$\Longleftarrow\overline{[]}_{10}$

FIGURE 2.7. *The failure of projection in universe with ME chords of Class III.*

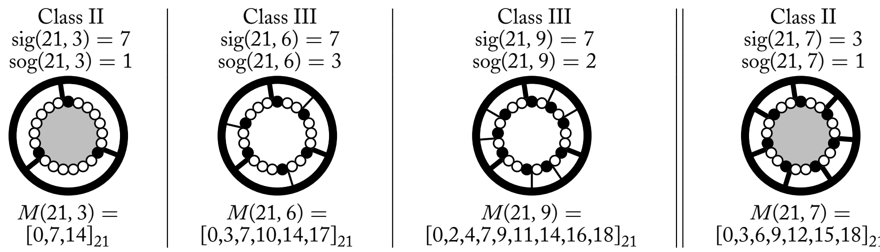


FIGURE 2.8. *Intervallic Half-Truth counterexamples from the 21-pc universe*

and the genera's index numbers is not a coincidence. As an exercise, the reader may confirm in connection with Figure 2.8 that the characteristic ME subgenera of $\mathbb{Q}(21, 3)$, $\mathbb{Q}(21, 6)$, and $\mathbb{Q}(21, 9)$ are all generated by projections of ic 7 (with different "shifts") and, reciprocally, that the characteristic ME subgenus of $\mathbb{Q}(21, 7)$ can be generated by projecting ics 3, 6, or 9. Further consideration of this issue would lead to some interesting generalizations about qualitative genera in other universes, but since our primary responsibility is to our own 12-pc universe, we will abandon that particular line of thought.

Suffice it to say that we have now established the sense in which the Intervallic Half-Truth is half false. In the larger metatheoretical context of our undertaking, this demonstration has enormous explanatory value. It explains why we do not have an agreed-upon methodology for using interval content to speak about chord quality; to be sure, there is some agreement that ic vectors give us information of that kind, but attempts to instantiate that notion in concrete theoretical constructs have led to a proliferation of ad-hoc methodologies grounded only in empty appeals to intuition. Our consideration of the Intervallic Half-Truth suggests that ic vectors model qualitative intuitions only by accident, as it were, and simply because our pitch-class universe has no Class III structures. (Norman Carey has conjectured in private correspondence that there are indeed no such universes larger than the usual one.)

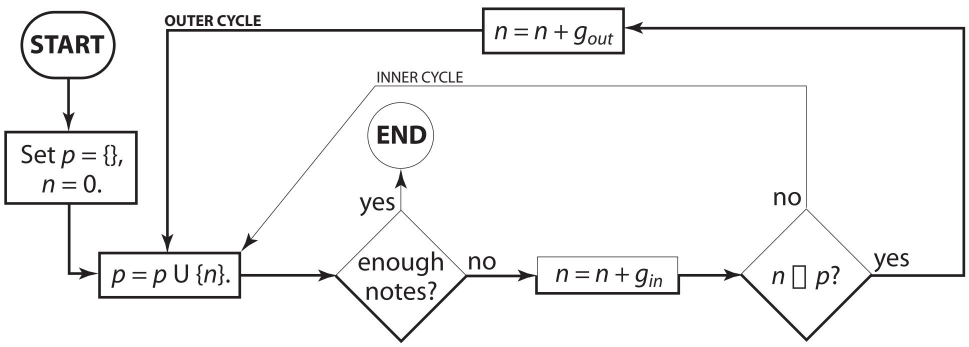


FIGURE 2.9. *Hanson's projection procedure generalized.*

§ 2.3 Generalizing up to generic prototypes.

We have been speaking rather loosely about the idea that certain ME chords are "doubly generated." It will be helpful to clear this notion up somewhat, since it turns out that this expanded sense of *generated* applies to all ME chords, and also connects helpfully to Hanson's projection procedure.

Figure 2.9 shows a slightly more general version of Hanson's projection procedure (compare Figure 1.1). In this version, Hanson's fixed "shift" of a semitone (or, originally, a fifth), is replaced by a variable **outer generator** (g_{out}). What was previously conceptualized as "the interval being projected" is now called the **inner generator** (g_{in}). It is asserted without proof that every ME chord (even if it is not Class I or IIa, and therefore generated in the traditional sense) can be generated by this procedure. For example, the Class IIb $M(12, 8)$, an octatonic collection, can be generated with $g_{\text{in}} = 3$ and $g_{\text{out}} = 1$, and the Class III $M(10, 4)$ can be generated with $g_{\text{in}} = 5$ and $g_{\text{out}} = 2$. We will now give formulas for the smallest possible g_{in} and g_{out} that will generate an exemplar of a given ME species.

The **smallest inner generator** of $M(c, d)$ — symbolized $\text{sig}(c, d)$ — is equal to $c/\text{gcf}(c, d)$. This is equal to the index, or number of distinct exemplars, of $M(c, d)$. It is also equal to the size of the smallest subuniverse in which into which $M(c, d)$ can be projected down by a CYCLE homomorphism in the manner of Figures 2.5 and 2.6. Certain additional features of sig(c, d) depend on the classification of $M(c, d)$:

Class I For the same reason these ME chords are not TINV — namely, that gcf(c, d) $= 1$ — it is the case that sig(c, d) $= c$. Since $c \equiv 0 \mod c$, the inner generator *qua* generator effectively does not exist, and we will see that these chords are generated entirely by their outer generators.

Class IIa In this case, gcf(c, d) $= d$, so sig(c, d) $= c/d$. This is an interval precisely one *d*th the size of the universe, so the process of generating Class IIa ME chords stops just at the point that it would reach the outer cycle. In contrast to Class I ME chords, then, these are entirely generated by their *inner* generators.

Class IIb Since gcf(c, d) $=$ gcf($c, c - d$), the smallest inner generator of a Class IIb $M(c, d)$ is equal to that of its complement, a Class IIa chord. This allows us to synthesize the notion that these chords are CG with the notion of "smudging" — we see that the smudge is provided by the outer generator in this case.

The **smallest outer generator** of $M(c, d)$ — symbolized sog(c, d) — is the smallest positive integer s such that $s \cdot d/\text{gcf}(c, d) \equiv \pm 1 \mod \text{sig}(c, d)$. This an unwieldy way of saying something rather simple: suppose we project $M(c, d)$ down into the smallest subuniverse that preserves the structure of its repeated intervallic pattern; as mentioned above, the size of this subuniverse is sig(c, d), and the resulting projected chord is a Class I ME chord — of which sog(c, d) is the smallest generator. An example is probably in order. $M(12, 8)$, the octatonic species, can be projected down into a three-pc universe as the species $[01]_3$, which preserves the 1–2 pattern of semitones. Recall, in connection with our discussion of Cohn's CYCLE homomorphisms (§ 1.3.3), that this projection can be conceptualized in terms of minor-third (rather than octave) equivalence. Then sog(12, 8) is the smallest positive integer s such that $s \cdot 8/4 \equiv \pm 1 \mod 3$. This is equal to 1, since $1 \cdot 2 \equiv -1 \mod 3$, and 1 is surely a generator of $[01]_3$. Again, Class I and II ME species have certain special properties:

Class I Since $\text{gcf}(c, d) = 1$ and $\text{sig}(c, d) = c$, everything becomes much simpler: $\text{sog}(c, d)$ is the smallest positive integer which, multiplied with d, is equal to ± 1 mod c. In other words, $\text{sig}(c, d)$ is a generator of $M(c, d)$. Recall our earlier claim "that these chords are generated entirely by their outer generators."

Class IIa Since Class IIa ME chords "are generated entirely by their *inner* generators," there is a sense in which their outer generators are vestigial. However, since $\text{gcf}(c, d) = d$, and since therefore the crucial term in the definition, $d/\text{gcf}(c, d)$, is equal to 1, everything becomes much simpler in this case too: $\text{sog}(c, d) = 1$.

Class IIb Again, we can use the fact that $\text{gcf}(c, d) = \text{gcf}(c, c-d)$ to establish that $\text{sog}(c, d) = 1$ by means of the argument for Class IIa cases. Here, however, the outer generator is not vestigial — in particular, the fact that it is 1 in this case now fully explains the "smudging" idea.

There are many interesting things to say about the smallest inner and outer generators of Class III ME chords, but they would be an unnecessary distraction from our immediate goal. As an exercise in the foregoing definitions, however, it is worthwhile to check in connection with Figure 2.8 that $\text{sig}(21, 9) = 7$ and $\text{sog}(21, 9) = 2$, and to study how that is played out in the structure of the Class III $M(21, 9)$ and its double generation as an inner TINV ic-7 cycle — itself $M(21, 3)$ — "projected" (in Hanson's sense) through an incomplete outer ic-2 cycle.

We now return to our theory of generic prototypes. So far, we have the notion that in a c-pc universe there is one qualitative genus $\mathbb{Q}(c, q)$ for every $0 < q \leq c/2$, that its prototypes include the species of its characteristic ME subgenus $\mathbb{M}(c, q)$, and that it may be characterized as Class I, II, or III, like $\mathbb{M}(c, q)$, by virtue of $\text{gcf}(c, q)$. Now that we have generalized the structure of ME chords in terms of the idea of double generation, we are in a position to circumscribe the **primary prototypes** of $\mathbb{Q}(c, q)$:

Class I Generalizing from the fact that the exemplars of the characteristic ME subgenus $\mathbb{M}(c, q)$ are not TINV, but are generated — in the simpler traditional sense — by $\mathrm{sog}(c, q)$, we assert that any species generated by $\mathrm{sog}(c, q)$ is a primary prototype of $\mathbb{Q}(c, q)$.

Class II and III Generalizing from the fact that the exemplars of the characteristic ME subgenus $\mathbb{M}(c, q)$ are TINV and doubly generated (trivially so in the case of Class IIa ME species), we assert that any TINV species doubly generated by $g_{\mathrm{in}} = \mathrm{sig}(c, q)$ and $g_{\mathrm{out}} = \mathrm{sog}(c, q)$ is a primary prototype of $\mathbb{Q}(c, q)$. These are all and only the species with those generators whose cardinality is a multiple of $\mathrm{gcf}(c, q)$.

Separate definitions according to generic classification are actually unnecessary. Since, for Class I genera, $\mathrm{gcf}(c, q) = 1$, we can unify the foregoing definitions as follows: a species is a primary prototype of $\mathbb{Q}(c, q)$, irrespective of the genus's class, if and only if

(DG) it is doubly generated by $g_{\mathrm{in}} = \mathrm{sig}(c, q)$ and $g_{\mathrm{out}} = \mathrm{sog}(c, q)$; and

(CC) its cardinality is a multiple of $\mathrm{gcf}(c, q)$, meaning that all inner cycles of the generation process are "complete."

Every species in $\mathbb{Q}(c, q)$'s characteristic ME subgenus is also a primary prototype.

To see how far we have come, and how far we still have to go, let us check our work against Figures 1.2 and 2.7, in which primary prototypes are labeled **1**. With respect to Class I genera, we have covered all our bases — but in a certain respect this is no great achievement, since the traditional notion of *generated* handles these cases already. With respect to Class II and III genera, we have dealt with all and only ME and TINV cases, indicated graphically with thick circles and gray interior shading, respectively. There are, however, Hanson projections that are neither TINV nor ME, e.g., 3–5 [016], which we would like to be a prototype of $\mathbb{Q}(12, 2)$. This can be accomplished by defining a **secondary prototype** of $\mathbb{Q}(c, q)$ as a species with DG (but not necessarily CC). By lifting condition CC, which derives from the TINV restriction on the particular definition of primary prototypes for Class II and III genera, we regain all the Hanson projections. (This follows *a fortiori* from the fact that the double-generation principle is a generalization of Hanson's projection principle.) All primary prototypes are also secondary prototypes. Secondary prototypes in Figures 1.2 and 2.7 that are not also primary prototypes are labeled **2**.

Our original discussion of Figure 2.7 led to the conclusion that Hanson projections fail to tell the whole story in all universes. The Intervallic Half-Truth derives in part from the fact that uniqueness of generators (up to interval-class) for nontrivial ME species is a special feature of the twelve-pc universe — our counterexample was $M(10, 5) = [02468]_{10}$, which is generated both by ic 2 and ic 4. To be sure, $\mathrm{sig}(10, 5) = 2$, but this is only the *smallest* inner generator of $[02468]_{10}$, and we must not allow the Intervallic Half-Truth to tempt us to ignore the fact that ic 4 is equally a generator of this species. While lifting CC generalizes our characterization of ME species in a way that would give us the Hanson projections in the twelve-pc universe, this move fails to generalize to all relevant Hanson projections in universes with Class III genera. Simply put, lifting CC would cause DG to manifest the Intervallic Half-Truth by saying, e.g., that $[024]_{10}$ is a prototype of $\mathbb{Q}(10, 5)$ while $[026]_{10}$ is not.

It is necessary, therefore, to seek yet another generalization that will expand us outward from the secondary prototypes. We start by observing that within any qualitative genus, all secondary prototypes are Kh-related to one another, which stems from two other properties. First, all of the secondary prototypes of a genus $\mathbb{Q}(c, q)$ can be generated by the same pair of generators — $\mathrm{sig}(c, q)$ and $\mathrm{sog}(c, q)$ — allowing us to interpret a smaller prototype as an early stage in the generation of a larger one, which therefore includes the smaller. Second, as it is possible to prove by an argument tangential to our purposes, the primary prototypes have what we referred to earlier as PCP: the complement of a secondary prototype is also a secondary prototype. Taken together, these two properties entail that all secondary prototypes of a given genus are mutually Kh-related.

Suppose, then, we generalize the foregoing observations by simply saying that a species is a **tertiary prototype** of $\mathbb{Q}(c, q)$ if and only if it is Kh-related to all of the genus's primary (but not necessarily secondary) prototypes. This is equivalent, by virtue of the argument given in the previous paragraph, to saying of a species that

> **(SW)** it is sandwiched (Lewin, 1987) between the next smaller and next larger primary prototypes; that is, it includes the former and is included in the latter.

This allows us to relax the TINV condition on Class II and III primary prototypes (CC), while simultaneously relaxing the restriction on generators (DG) that we saw as a manifestation of the Intervallic Half-Truth in the general case. In other words, all species with DG (whether or not they have CC) necessarily have SW, but the converse is not true; not all species with SW necessarily have DG. A familiar example concerns the prototypes of $\mathbb{Q}(12, 4)$, for which $\gcd(c, q) = 4$, meaning that the primary prototypes have multiples of four notes: the empty chord, the diminished-seventh chord, the octatonic collection, and the aggregate. Consider now the hexachordal prototypes of this genus. Hanson's hexachordal ic-3 projection is the secondary prototype 6–27 [013469], which has DG but not CC; it also has SW. The only other hexachordal species with SW in this genus is the Petrushka chord, the tertiary prototype 6–30 [013679], which has neither DG (it is not by any means doubly generated) nor CC. This species came up a number of times in Chapter 1, in connection with Eriksson's intervallic approach, Morris's algebraic approach, and Cohn's cyclic approach.

All secondary prototypes are also tertiary prototypes; tertiary prototypes in Figure 2.7 that are not also secondary (and, *a fortiori*, primary) prototypes are labeled **3**. It should be clear that we have now solved the problem presented by the Intervallic Half-Truth; all of the species listed under the heading of $\mathbb{Q}(10, 5)$ are now accounted for as tertiary prototypes. None of the other qualitative genera in the ten-pc universe has tertiary prototypes, and, conversely, no tertiary prototype of $\mathbb{Q}(10, 5)$ is missing from Figure 2.7. The entire collection of prototypes of $\mathbb{Q}(10, 5)$ can now be simply characterized as those species that include or are included in its characteristic ME subgenus $\mathbb{M}(10, 5)$. It should be pointed out immediately that not every system of generic prototypes can be characterized so simply; for example, the species $[02]_{10}$ is not included among the prototypes of $\mathbb{Q}(10, 4)$, even though the species is included in $\mathbb{M}(10, 4)$. Only qualitative genera of the form $\mathbb{Q}(c, c/2)$ or $\mathbb{Q}(c, c/3)$ have prototypes that admit of such a simple characterization, since those genera have the special property that the family of nontrivial primary prototypes is equal to the characteristic ME subgenus, which (since every chord is Kh-related to the empty chord and the aggregate) causes SW to determine nothing more than the Kh-complex about the characteristic ME subgenus.

Figure 1.2 lists no tertiary prototypes, since it originated in a discussion of Hanson projections that took place before the Intervallic Half-Truth was on the table. Figure 2.10 lists all tertiary prototypes in the twelve-pc universe, together with the other prototypes of their genera; they are limited to the genera $\mathbb{Q}(12, 4)$ and $\mathbb{Q}(12, 6)$. In the latter case, we observe that the prototypes of $\mathbb{Q}(12, 6)$ are exactly those species including or included in (i.e., the Kh-complex about) the whole-tone collection, which phenomenon is explained by the reasoning given in the previous paragraph. The former case has already been dealt with in our discussion of the Petrushka chord; we note, furthermore, that the tritone and its complement are also tertiary prototypes of $\mathbb{Q}(12, 4)$. Readers uncomfortable with this notion are gently reminded of the Intervallic Half-Truth. The key notion is that the essential qualitative properties of $\mathbb{Q}(12, 4)$ are embodied in the characteristic ME subgenus — including the octatonic collection, in which the role played by the interval of a tritone is no more or less essential than that played by the minor third. Again, by the argument from the previous paragraph, we note that the prototypes of $\mathbb{Q}(12, 4)$ are exactly the Kh-complex about the characteristic ME subgenus.

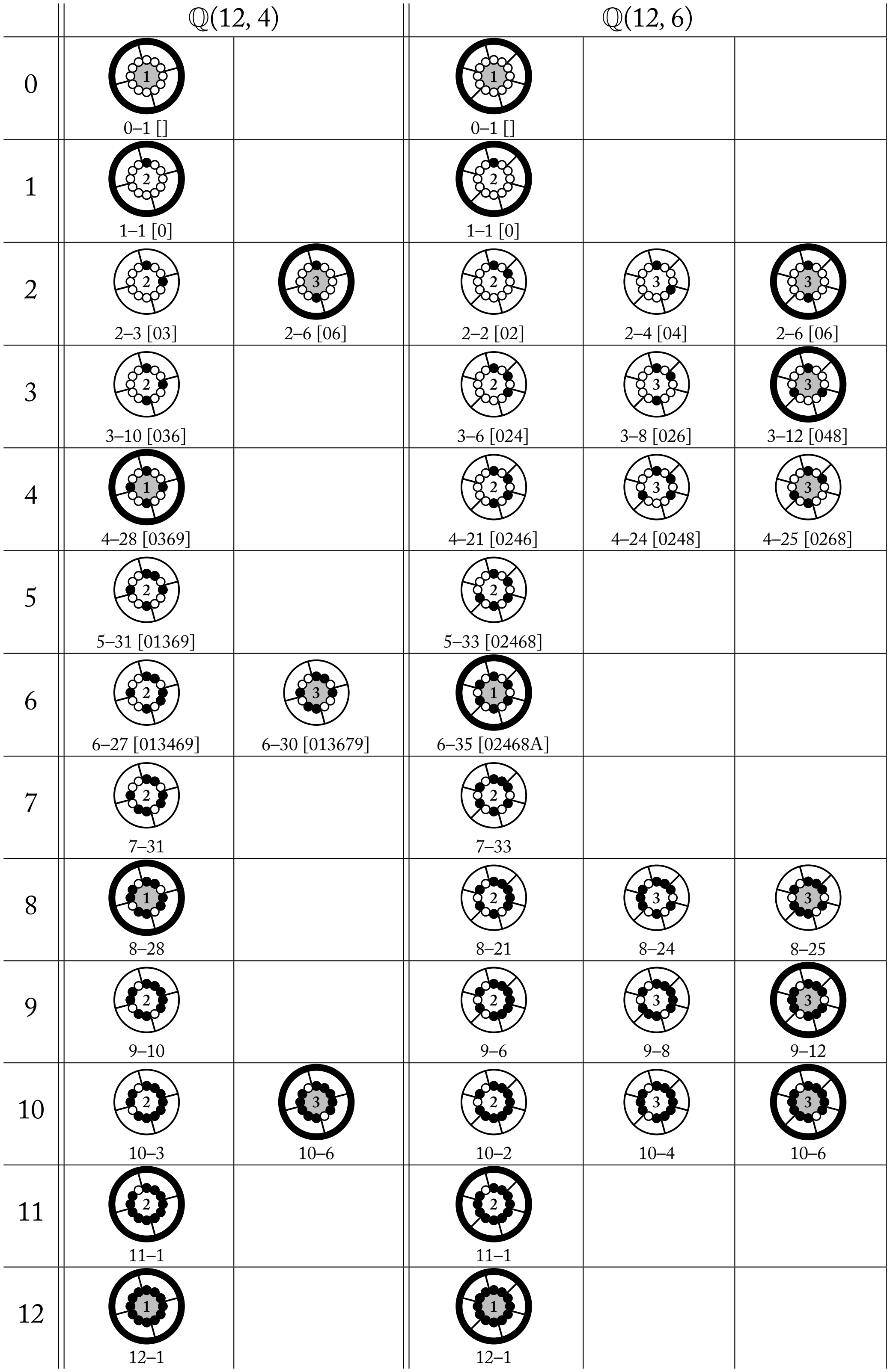


FIGURE 2.10. *Tertiary prototypes in the twelve-pc universe.*

§ 2.4 On theoretical unification

We have now completed the first stage in constructing a unified theory of chord quality, having specified necessary and sufficient conditions for qualitative genera and their prototypes in every equal-tempered pitch-class universe. Before extending our investigation to the theory of affinities, we pause to summarize our work so far, and to reflect on some metatheoretical implications of the approach.

2.4.1 The story so far. At the heart of every qualitative genus is not an interval class, but a family of ME species — its characteristic ME subgenus. With respect to the prototypical species of the characteristic ME subgenus, each qualitative genus has what we called UPP in Chapter 1, since there exists at least one ME prototype in each genus; it also has UGP, since each ME species belongs to the characteristic ME subgenus of exactly one genus.

Our classification of ME species, initially characterized in terms of G, TINV, and complementation, allowed us to generalize the prototypicality of the ME subgenus to a family of primary prototypes, all of which are either G (for Class I genera) or TINV (for Class II and III genera). A generalization of Hanson's projection procedure led to a sketch of the new result that ME chords of every class are doubly generated; this, in turn, allowed us to generalize primary-prototype properties up to the definition of the family of secondary prototypes. In the usual pc universe, the secondary prototypes (under which heading we mean to include the primary prototypes and the characteristic ME subgenus as well) coincide precisely with the Hanson projections. Noting the problem raised by the Intervallic Half-Truth in other universes, we expanded the prototypes once again, by means of the inclusional notion of the Kh relation, to the family of tertiary prototypes (which include secondary and primary prototypes as well as the characteristic ME subgenus).

What makes this a unified theory of generic prototypes is the extent to which it depends on all of the different approaches discussed in Chapter 1, together with the core concepts of maximal evenness and well-formedness from the scale-theoretic approach. This latter is not normally associated with the qualitative aspect of pcset theory, although a precedent for this work is to be found in Clough et al. (1999). Our generalizations from characteristic ME subgenera to primary and secondary prototypes relied both on Hanson's intervallic approach and Cohn's cyclic approach to transpositional invariance; the final generalization to tertiary prototypes relied entirely on Forte's inclusional approach.

We will see in the next chapter that everything we have done is grounded in a fundamentally algebraic approach due to David Lewin. His approach admits of a new interpretation that neatly fuzzifies all of the foregoing work directly into a theory of affinities, which does not depend on "external" similarity relations, and which is cast in vastly simpler terms than we have been able to manage in this chapter. The eclectic constitution of our theory in its present state, however, is not an artifice, but a necessary reflection of the data it seeks ultimately to explain — viz., the widespread agreement to be observed among the divergent approaches we have considered. In taking the time to show rather precisely how the approaches are interconnected at the fundamental level of prototypes, we have been able to ensure that our generalizing move into the theory of affinities will take all of the individual approaches with it at once.

2.4.2 A non-intervallic characterization of interval content. As a preview of what is to come, we will now demonstrate one aspect of the unified nature of our theory by showing that it is possible to characterize the interval content of any primary prototype of $\mathbb{Q}(c, q)$ quite generally in terms of q, c, and the prototype's cardinality, without actually counting intervals.

Consider the species p, a primary prototype of the qualitative genus $\mathbb{Q}(c, q)$. We have established that d (the cardinality of p) is a multiple of gcf(c, q); because an appropriate generalization of the Complement Theorem establishes a relationship between the interval content of a chord and that of its complement, we may assume without loss of generality that $d \leq c/2$. Let the notation $|x|_c$ mean "the least nonnegative integer

$i\ (= \lvert qi\rvert_c)$	0	1	2	3	4	5	6	
$\max(2 - \lvert qi\rvert_c, 0)$	2	1	0	0	0	0	0	← ic vector of [01]
$\max(3 - \lvert qi\rvert_c, 0)$	3	2	1	0	0	0	0	← ic vector of [012]
$\max(4 - \lvert qi\rvert_c, 0)$	4	3	2	1	0	0	0	← ic vector of [0123]
$\max(5 - \lvert qi\rvert_c, 0)$	5	4	3	2	1	0	0	← ic vector of [01234]
$\max(6 - \lvert qi\rvert_c, 0)$	6	5	4	3	2	1	0	← ic vector of [012345]

FIGURE 2.11. *Interval content of primary* $\mathbb{Q}(12, 1)$ *prototypes.*

i	0	1	2	3	4	5	6	7	8	9	10	11
$5i$	0	5	10	15	20	25	30	35	40	45	50	55
$\lvert qi\rvert_c\ (= \lvert 5i\rvert_{12})$	0	5	2	3	4	1	6	1	4	3	2	5
$\max(0, 2 - \lvert qi\rvert_c)$	2	0	0	0	0	1	0	← ic vector of [05]				
$\max(0, 3 - \lvert qi\rvert_c)$	3	0	1	0	0	2	0	← ic vector of [027]				
$\max(0, 4 - \lvert qi\rvert_c)$	4	0	2	1	0	3	0	← ic vector of [0257]				
$\max(0, 5 - \lvert qi\rvert_c)$	5	0	3	2	1	4	0	← ic vector of [02479]				
$\max(0, 6 - \lvert qi\rvert_c)$	6	1	4	3	2	5	0	← ic vector of [024579]				

FIGURE 2.12. *Interval content of primary* $\mathbb{Q}(12, 5)$ *prototypes.*

congruent to $\pm x$ modulo c." This is an operation we are accustomed to performing when naming interval classes; 5, for example, is the least nonnegative integer congruent to ± 7 modulo 12, so $\lvert 7\rvert_{12} = 5$.

Here is the general statement: for $0 \leq i \leq c/2$, the number of instances of ic i in p is equal to $(d - \lvert qi\rvert_c)$, or, if this figure is negative, to 0. Figure 2.11 shows how this works in the simple case of $\mathbb{Q}(12, 1)$ — for a related, if less general, figure and discussion, see Carey (1998, pp. 29, 280). Under our working assumptions about i, $\lvert qi\rvert_c = i$ in this case, since $q = 1$. The "floor" of 0 creates the impression, as d increases, of the ic vector of 6–1 [012345] rising up from below the surface — of a constant shape whose profile is increasingly revealed as d approaches 6. Such a shape is what presumably is being described by Eriksson's generic "models," or compared by intervallic similarity relations.

Figure 2.12 shows the analogous process for the case $\mathbb{Q}(12, 5)$ and breaks the move from i to $\lvert qi\rvert_c$ into two steps, extending the calculation to all values of i to exemplify the number-theoretic principle that if i and j belong to the same interval class (i.e., if $\lvert i\rvert_c = \lvert j\rvert_c$), then necessarily $\lvert qi\rvert_c = \lvert qj\rvert_c$. The effect of taking $\lvert i\rvert_{12}$ to $\lvert 5i\rvert_{12}$ is evidently a permutation that simply swaps interval-classes 1 and 5. This permutation, applied at the abstract level of the ic-vector profile that emerges as d approaches 0, is exactly what relates the individual ic vectors of analogous prototypes of $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$. Note that this resembles the establishment of intergeneric affinities between these two genera by the multiplicative M_5 (as discussed extensively in Chapter 1), except for the fact that we are not dealing directly with chords or even pitch-classes here — only making predictions about them on the basis of an abstract sort of "interval-class multiplication."

Figure 2.13 extends the calculation to the remaining primary prototypes in the twelve-pc universe. The principle mentioned above, that if $|i|_c = |j|_c$, then $|qi|_c = |qj|_c$, continues to hold; we see now that this does not necessarily work in reverse, since this "interval-class multiplication" seems to have an epimorphic (many-to-one) character in the manner of Cohn's CYCLE-*n* homomorphisms — when $q = 3$, for example, we see that for all odd i, $|qi|_{12} = 3$. Technical matters aside, however, what is of primary interest is the fact that this same formula — which works by multiplying interval-classes by index numbers of qualitative genera — accurately predicts the interval vectors of all qualitative prototypes. (Note that each tritone is counted as two instances of ic 6, a familiar enough circumstance in pcset theory for us to leave it as is.) Furthermore, it continues to show a consistency of profile among prototypes of the same genus. This is a feature built into the formula itself, which establishes that among any two primary prototypes of the same genus, the difference in ic-i content is only a function of cardinality and the "masking" effect of the floor at 0. We should not be surprised, therefore, to revisit Figure 1.6 and find that Eriksson's generic models of ic content correspond one to one with the models produced by our formula as q varies from 1 to 6.

These findings, which pertain directly only to primary prototypes, can be extended to all prototypes. The "sandwiching" property (SW), which is the sole necessary and sufficient condition on the broadest construal of prototypicality, tightly circumscribes the interval content of prototypes that are not themselves primary. Thus while our formula does not directly predict the interval content of all prototypes, it describes strong constraints on interval content — the very same constraints Eriksson described by means of his models.

	i	0	1	2	3	4	5	6	7	8	9	10	11
$q = 2:$	qi	0	2	4	6	8	10	12	14	16	18	20	22
	$\lvert qi\rvert_c$ $(= \lvert 2i\rvert_{12})$	0	2	4	6	4	2	0	2	4	6	4	2
	$\max(0, 2 - \lvert qi\rvert_c)$	2	0	0	0	0	0	2	← ic vector of [06]				
	$\max(0, 4 - \lvert qi\rvert_c)$	4	2	0	0	0	2	4	← ic vector of [0167]				
	$\max(0, 6 - \lvert qi\rvert_c)$	6	4	2	0	2	4	6	← ic vector of [012678]				
$q = 3:$	qi	0	3	6	9	12	15	18	21	24	27	30	33
	$\lvert qi\rvert_c$ $(= \lvert 3i\rvert_{12})$	0	3	6	3	0	3	6	3	0	3	6	3
	$\max(0, 3 - \lvert qi\rvert_c)$	3	0	0	0	3	0	0	← ic vector of [048]				
	$\max(0, 6 - \lvert qi\rvert_c)$	6	3	0	3	6	3	0	← ic vector of [014589]				
$q = 4:$	qi	0	4	8	12	16	20	24	28	32	36	40	44
	$\lvert qi\rvert_c$ $(= \lvert 4i\rvert_{12})$	0	4	4	0	4	4	0	4	4	0	0	4
	$\max(0, 4 - \lvert qi\rvert_c)$	4	0	0	4	0	0	4	← ic vector of [0369]				
$q = 6:$	qi	0	6	12	18	24	30	36	42	48	54	60	66
	$\lvert qi\rvert_c$ $(= \lvert 6i\rvert_{12})$	0	6	0	6	0	6	0	6	0	6	0	6
	$\max(0, 6 - \lvert qi\rvert_c)$	6	0	6	0	6	0	6	← ic vector of [02468A]				

FIGURE 2.13. *Interval content of other primary prototypes in the twelve-pc universe.*

2.4.3 Harmony and voice leading. The paradigm case of unification in a modern music-theoretical context is, of course, Schenkerian theory, which grows out of the influential and powerful idea that harmony and voice leading are two sides of the same coin. The application of this idea to repertoires outside of Schenker's restrictive canon is by no means limited to specifically Schenkerian approaches; Schoenberg himself, often characterized as Schenker's antithesis, proclaimed famously that "THE TWO-OR-MORE-DIMENSIONAL SPACE IN WHICH MUSICAL IDEAS ARE PRESENTED IS A UNIT" (Schoenberg, 1950, p. 109; emphasis original), and similar sentiments were common among postwar composers of the Darmstadt circle. The issue has recently been tackled from a pcset-theoretic perspective in several important articles (Roeder, 1994; Lewin, 1998; Morris, 1998; Straus, 2003). Straus begins his contribution with a clear statement of the theoretical problem: "Theories of atonal music have traditionally been better at describing harmonies — at devising schemes of classification and comparison — than at showing how one harmony moves to another" (p. 305). The "schemes of classification and comparison" include the variously taxonomic approaches discussed in Chapter 1 in connection with the notion of chord quality. These are theories of chord structure, and as Straus points out later in the same article,

> the internal structure of a set shapes the kinds of voice leading connections it can create with other sets. And the reverse is also true: Voice leading constrains harmony in the sense that the kinds of voice leading connections a set is capable of making with other sets will go a long way toward defining its internal structure" (p. 341).

Straus styles his approach to voice leading as transformational; the idea is that the assertion of species identity among two different chords (under transpositional and inversional equivalence) implies a natural voice-leading mapping from one to the other. This is not the place to go into detail about any theory of voice leading, but we are in a position to make some highly general remarks about the way in which chord structure, as we are coming to understand it, informs the kind of voice-leading connections it can make. Consider, for example, any primary prototype of a Class II genus; it is necessarily invariant under some inversion and some nontrivial transposition. Now suppose there are two identical instances of it, and we wish to make transformational voice-leading connections between them. How do we proceed? Are we simply to connect each pitch-class in one chord to the identical pitch-class to the other? Or are we to imagine that the chords are transpositions of one another, or possibly inversions? The answer, of course, depends on context.

The notation in Figure 2.14 analyzes the first few bars of the seventh movement of Messiaen's *Quatuor pour la fin du temps*, which is scored for piano and cello. The piano part, which in the original consists of chords in constant sixteenth notes, has been registrally normalized and rhythmically simplified in the lower staff, and the cello part appears with Messiaen's idiosyncratic slurs in the upper staff; for the sake of clarity, some liberty has been taken with enharmonic spelling. All of the notes in the passage belong either to {0369} or to {147A}; round noteheads indicate the former, and triangular noteheads the latter. Both of these chords belong to the same species, a primary prototype of $\mathbb{Q}(12, 4)$. The passage shown, which suggestively lacks any note in the other chord of the same species, {258B}, contains two three-bar phrases and the beginning of an extended third phrase. System breaks in the example separate phrases.

FIGURE 2.14. *Analysis of the first few bars of Messiaen's "Quatuor," vii.*

The second phrase (beginning in bar 4) is a literal transposition of the first down a minor third, and a somewhat altered minor-third transposition yields the third phrase (beginning in bar 7)from the second. But because the total pitch content of all three phrases is the C/C♯ octatonic collection, a strong harmonic continuity effaces the transformation that has taken place. This continuity is particularly evident in the cello line, which comprises five pitch-classes in each phrase, four of which belong to a single [0369] chord. Under the transposition in question, these four pitch-classes transform into one another, and only the "odd note out" (which has a triangular notehead in each case) is altered.

Messiaen handles his harmonic material with great subtlety. The piano plays the complete octatonic collection in each phrase, and since the collection is invariant under the transpositions employed, the general harmonic field is static throughout the passage; the pulsating and registrally refracted texture of the actual piano part (not shown) helps to realize the potential for stasis and continuity. Moreover, the greater part of the cello line — the last two-thirds of each phrase — avoids the odd note, so as to dovetail the harmonic content of each phrase into the next. Yet by beginning each phrase with its

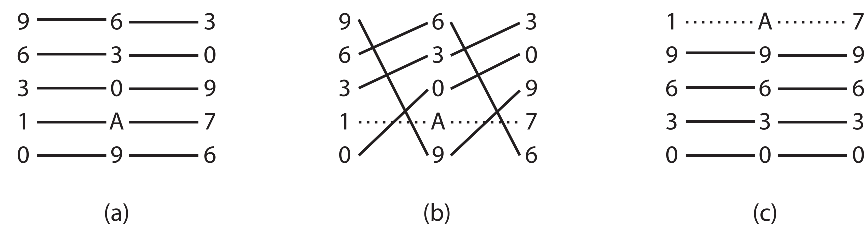


FIGURE 2.15. *Transformational voice-leading ambiguity among Messiaen's cello lines.*

unique odd note, and by quickly reiterating that note, Messiaen clearly demarcates the beginning of each phrase by means of what limited harmonic contrast his material and technique makes available.

The analysis we have just undertaken has at its heart a claim about the passage's dialectical negotiation of harmonic stasis and harmonic articulation — the notion that the pentachordal cello lines each have an "odd note out," which provides the articulation, depends on that note standing apart from the very same TINV tetrachordal subset that provides the stasis. In terms of transformational voice leading, we would model the idea of harmonic articulation by asserting voice-leading connections among the notes of the cello line so that each voice descends by three semitones across each phrase boundary, and the idea of harmonic stasis by connecting common tone to common tone, with only the "odd note out" moving. This latter model, in which the voices do not move in concert to delineate a transposition or inversion, reflects what Straus calls *fuzzy transposition,* a term he attributes to me (1996). The technical foundation of the analysis, then, is an ambiguity between nonfuzzy transposition through three semitones (Figure 2.15a) and fuzzy "transposition" through no semitones (Figure 2.15b and c). Clearly this ambiguity, or something like it, is enabled by the fact that the chord involved is a secondary prototype of $\mathbb{Q}(12, 4)$. The primary prototypes of this genus figure into our analysis as well, and for them the ambiguity between a minor-third transposition and a trivial "transposition" is total; there is no fuzziness in either reading.

The general point, which deserves a complete and separate study, relates to the fact that the voice-leading behavior of the chords in question is characteristic of their genus. Similar kinds of ambiguities are inherent in (and therefore characteristic of) prototypes of other genera. Our treatment of the relationship between a genus and the constraints on its prototypes' interval content reinforces the point — thanks to the so-called Common-Tone Theorem, a chord's interval content determines an aspect of its transformational voice-leading behavior, viz. the extent to which any given transposition yields common tones and thereby comes closer to resembling non-transposition. Once a theory of generic affinities is in place, we would expect it to predict "the kinds of voice leading connections a set is capable of making" insofar as the characteristic voice-leading behavior of a given genus is reflected (to a certain degree) in any chord having a strong affinity to the genus.

CHAPTER 3

A generalized theory of affinities

Our goal is, in principle, to fuzzify the notion of prototypes developed in the last chapter, so that prototypicality in a qualitative genus becomes the limit case of a property that comes in degrees; this will give us a theoretical framework that provides, for every pitch-class universe, a set of graded categories with a radial structure (generic prototypes together with intrageneric affinities) of the type discussed in Chapter 1. We will begin by clearing our theoretical slate once again, starting with some new core concepts, and shortly thereafter we will find ourselves meeting up unexpectedly with the theory of prototypes. As we proceed, principles of intergeneric affinity will emerge directly from the new core concepts of the theory, although this will not be our focus.

Our point of departure is a pair of highly technical articles by David Lewin concerning the interval function (1959; 2001) — the first and last publications to appear during Lewin's lifetime. In the first of these, which is in a sense the earliest example of modern pcset theory, and in any case predates the concept of the pcset *per se*, Lewin introduces the interval function as a construct that serves, in a sense, as a very general way of talking about the voice leading between two pcsets, or what he later (1960) called the "total potential counterpoint" between pcsets. The basic question of the 1959 article, in simpler and more current language than Lewin had at his disposal at the time, is this: Suppose there are two pcsets p and q. Can we be certain that q is the only pcset that creates that particular "total potential counterpoint," as described by the interval function from p to q? The answer is yes, but there are some exceptions, and Lewin describes systematically what those exceptions are, offering tantalizingly brief sketches of a proof.

We can read Lewin's question as probing deeply into the problem of the relationship between harmony and voice leading long before the topic became current in pcset theory; the considerations elaborated at the end of Chapter 2 suggest that his solution to the problem may be of great importance. Assuming that the interval function, which was the basis of my earlier approach to fuzzy transposition (1996), truly does belong to the sphere of voice leading, and recalling that the interval function between p and q is subject to predictable symmetries under the transposition or inversion of q relative to p, the success of Lewin's answer to the question says something significant about the degree to which harmony and voice leading are intertwined. If we know the taxonomic properties of a known harmony p and the abstract total potential counterpoint between p and some unknown harmony q, we know at least a little bit about the taxonomic properties — the qualitative properties — of q as well.

§ **3.1** Fourier balances

We will now explore a way in which a remarkably simple and intuitive theory of chord quality can be teased out of Lewin's mathematical approach to the problem just described. Lewin provides a solution that involves the assertion of five properties that may apply to a chord; in the later article he refers to them collectively as the **Fourier properties**. While we need not be specifically interested in the uniqueness problem, what Lewin proves is that, provided q has at least the same Fourier properties that p has (and possibly others), then no other chord has the same "total potential counterpoint" with p that q does.

3.1.1 Lewin's five Fourier Properties. We begin with a review of the five Fourier properties, developing along the way a standpoint from which they can all be reduced to a single principle. All definitions quoted below are from the later article (2001, pp. 5–6).

1959	2001
Whole-tone-scale property	FOURPROP(6)
Diminished-seventh-chord property	FOURPROP(4)
Augmented-triad property	FOURPROP(3)
Tritone property	FOURPROP(2)
Exceptional property	FOURPROP(1)

FIGURE 3.1. *Lewin's 1959 and 2001 names for the Fourier properties.*

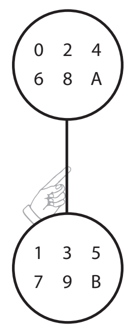


FIGURE 3.2. *Fourier Balance 6.*

The table in Figure 3.1 collates the names Lewin gave his properties in the earlier and later articles; four of the properties were named after familiar types of collections in the earlier article. It is crucial at the outset to be aware of two important facts: first, each of these collections belongs to the characteristic ME species of a qualitative genus (as theorized in the previous chapter); second, and counterintuitively, no such collection has the property named after it.

> **Fourier Property 6.** Also known as the **whole-tone-scale property**. A chord has this property if it "has the same number of notes in one whole-tone set, as it has in the other."

A useful way to think of the Fourier properties generally is that each invokes the notion of balance, some more abstractly than others. In this case, the property inheres in chords whose constituent pcs are balanced between the two whole-tone collections; between those that are even and those that are odd. Figure 3.2 suggests a somewhat literal way to imagine this: a balance scale consisting of two pans connected by a rigid beam, supported in the center of the beam. (We will refer to each such construction as a **Fourier balance**, numbering each according to the Fourier property it illustrates; thus Figure 3.2 schematically represents Fourier Balance 6.)

To determine if a given chord has Fourier Property 6, one might place a unit weight on one of the pans of the balance for each pc in the chord, according to whether the pc in question belongs to the even or odd whole-tone collection. If, after all pcs in the chord are accounted for, the balance is even, the chord has Fourier Property 6. From this point of view, it is easy to see why the whole-tone scales themselves do not have this property; no other kind of chord can tip the balance more!

Any transposition either transports all the pcs in one pan into the other or leaves them all where they are; so also for any inversion. As a consequence, if a chord has Fourier Property 6, all of its transposed and inverted forms also have the property, and so we can speak of the property as attaching the the species as a whole. Related observations can be made about all of the Fourier properties; although we will not do so here, the reader is invited to think about how transposition and inversion get played out on the other Fourier balances, as this is an excellent way to become acquainted with their structural properties.

> **Fourier Property 4.** Also known as the **diminished-seventh chord property**. A chord has this property if it "has the same number of notes in common with each of the three diminished-seventh chord sets."

This Fourier property is nearly as straightforward as Fourier Property 6. To have the property, a chord must have equal representation from all three tetrachords of the [0369] type. The associated Fourier balance (see Figure 3.3) has three pans and a three-armed support structure that balances on the point at which the arms intersect. We again call attention to the chord species that tip the balances the most — in this case either a diminished-seventh chord or its complement (an octatonic collection) will do it. Again, this is an instance of a general property — on any Fourier balance, the complement of a chord will tip the balance exactly as much as the chord, but in the

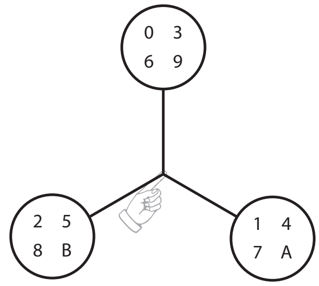


FIGURE 3.3. *Fourier Balance 4.*

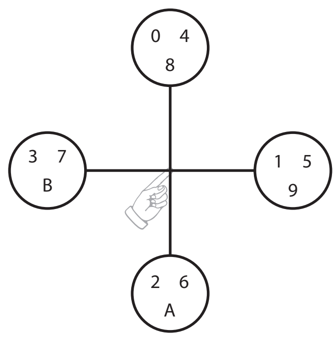


FIGURE 3.4. *Fourier Balance 3.*

opposite direction. From the point of view of the properties, this means that if a chord has some property, not only will all transposed and inverted forms of that chord have the property, but so will their complements.

> **Fourier Property 3.** Also known as the **augmented-triad property**. A chord has this property if, "for any augmented-triad set A, [it] has the same number of notes in common with T6(A), as it has in common with A."

One might expect that, by analogy with the two Fourier properties discussed so far, that equal representation from all four exemplars of the [048] species would form the basis of Fourier Property 3, but this is not the case. For the first time, the Fourier-balance model (see Figure 3.4) provides significantly more clarity than any verbal definition.

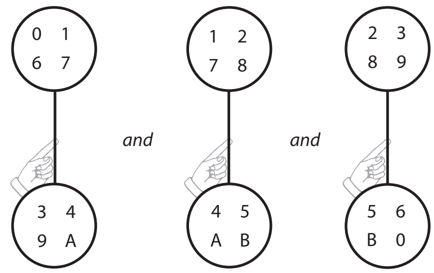


FIGURE 3.5. *Lewin's later model for Fourier Property 2.*

Here we have four pans affixed to a cruciform armature. Each pair of opposed pans corresponds to one of the two distinct [048] chords contained in a single whole-tone collection; for the whole contraption to be balanced, it is sufficient for the contents of each of these pairs of pans to be balanced. The chord {134567}, for example, has Fourier Property 3, since it has two pcs from each of the odd [048] chords, and one pc from each of the even [048] chords.

Maximum imbalance on Fourier Balance 3 is achieved not by augmented triads and their complements (each of which is a form of Messiaen's Mode 3) — although these produce a high degree of imbalance — but by the well-known hexatonic collections of Neo-Riemannian theory (Cohn, 1996).

> **Fourier Property 2.** Also known as the **tritone property**. A chord has this property if, "for any (0167)-set K, [it] has the same number of notes in common with T3(K), as it has in common with K."

The increasing difficulty of describing the Fourier properties continues here. Lewin actually gives different (but equivalent) definitions in the two articles; of the two, the later definition is more resonant with our methodology. It implies three separate balances, which are displayed in Figure 3.5; if (and only if) all three are simultaneously balanced under the weighting that models a chord, then that chord has Fourier Property 2. If it is peculiar that this property seems to require three independent balances, it is even more so that a duplicate set of weights is needed, since each pc corresponds to two of the six

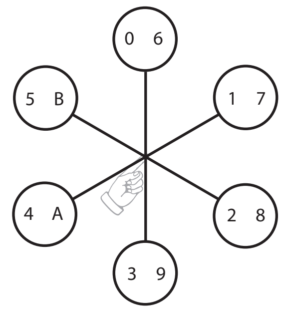


FIGURE 3.6. *Fourier Balance 2.*

pans in the set of balances. This makes it somewhat more difficult to construct chords with Fourier Property 2 using the Fourier-balance model. For example (and here, to prepare for a modest payoff in the next paragraph, the reader may wish to take a pencil to the figure and mark off pcs) the chord {0139} appears to produce equilibrium on the leftmost balance in Figure 3.5 — but of these four pcs, one is duplicated on the central balance and the other three are duplicated on the rightmost balance, and as a result, both of these balances will lean to one side.

Figure 3.6 shows a one-balance model for Fourier Property 2 that works just as well. The downside is that balance is slightly trickier to visualize on this model, because there are two separate ways in which balance can be achieved. Pairs of pitch-classes like 0 and 9 can balance each other out (when they correspond to opposite pans), but so can certain triples of pitch-classes like 1, 3, and 5 (when they correspond to pans equidistant from one another). Furthermore, in a chord such as {01359}, the two ways of balancing can operate simultaneously; this chord indeed has Fourier Property 2, which can be double-checked with reference to Lewin's definition and to a possibly marked-up Figure 3.5.

Let us distinguish between these two types of "atomic" equilibrium by saying that 0 and 9 are an **annihilating pair**, and that 1, 3, and 5 are an **annihilating triple**, noting the following important theorem: Any chord that has Fourier Property *n* can be partitioned into dyads and trichords that are annihilating pairs and triples, respectively, on Fourier

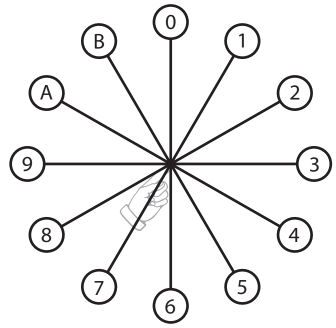


FIGURE 3.7. *Fourier Balance 1.*

Balance n. Of the Fourier properties studied up to now, only Property 2 pertains to chords that can be partitioned into subsets of both varieties — Property 6 and Property 3 both depend on annihilating pairs, and Property 4 depends on annihilating triples.

> **Fourier Property 1.** Also known as the **exceptional property**. A chord has this property if it "can be expressed as a disjoint union of tritone sets and/or augmented-triad sets."

This is the only one of Lewin's Fourier properties that he does not explicitly define in terms of a procedure for counting common tones; as we will see shortly, this is not the only way in which the property is exceptional in relation to the four we have studied already. However, his definition can be quite easily reworded into our working terminology as follows: Consider Fourier Balance 1 (see Figure 3.7), which looks remarkably like the usual pedagogical clockface; a chord has Fourier Property 1 just in case it can be partitioned into annihilating pairs ("tritone sets") and annihilating triples ("augmented-triad sets").

3.1.2 Completing and generalizing the system. The structure of each of the numbered Fourier balances can be completely and quite simply described, using clockface terminology, as follows: On Fourier Balance n, the pc p is located at $(n \times p)$ o'clock, where n and p are integers modulo 12 and should be multiplied accordingly. This statement about the interrelation of the Fourier balances (and their associated properties) is rather important, so we will give it a name: the *Multiplication Principle.*

We are now in a position to investigate a question thus far left unstated: Why are there only five Fourier properties, and why are they numbered 1, 2, 3, 4, and 6? Even without the Multiplication Principle, it may seem that Fourier Property 5 is missing; now one would do well to wonder about the other six integers modulo 12. The fact of the matter is that it is certainly possible to construct the full set of twelve Fourier balances — but with one trivial exception, any chord that is balanced on any of the twelve Fourier balances will be balanced on one of the five "canonical" Fourier balances we have studied. The trivial exception concerns Fourier Balance 0. According to the Multiplication Principle, all pcs are located at $0 = 12$ o'clock on Fourier Balance 0; since the fulcrum is still at the center of the clockface, only the null chord keeps this balance balanced. So while we may certainly assert a Fourier Property 0 that is independent of the others, its interest is largely theoretical. The balance itself, though, has the property that it tips in direct proportion to the cardinality of the chord placed on it, so it is maximally imbalanced under the aggregate itself.

Four of the remaining possible Fourier balances may be dispensed with by observing that Fourier Balance $(12 - n)$ is simply the image of Fourier Balance n in a mirror placed parallel to the 12 o'clock–6 o'clock axis. In this connection the symmetry of Fourier Balance 6 (and Fourier Balance 0) along that axis should be noted, as should the guarantee of this symmetry provided by the fact that these balances lie entirely along these axes. (Since the pans of the balances are metaphorical, we are free to suppose they are infinitesimal in size.)

That leaves us with Fourier Balances 5 and 7. Since they are related to each other by the mirror inversion described in the last paragraph, we need only consider one. Figure 3.8 displays Fourier Balance 5. Inspection reveals that, as was the case with Fourier Balance 1, all annihilating pairs are tritones, and all annihilating triples are of the [048] type. From this we may conclude that all and only chords with Fourier Property 1 will be balanced on Fourier Balance 5, and that there is therefore no need for a separate

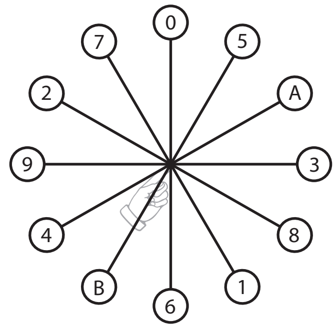


FIGURE 3.8. *Fourier Balance 5.*

Fourier Property 5. It is presumably the fact that this one property corresponds to two structurally distinct balances that caused Lewin to name it the "exceptional property" in 1959.

3.1.3 Fourier balances and qualitative genera. The reader will undoubtedly have noticed a connection between the Fourier balances and the qualitative genera that have been concerning us so far, particularly in connection with the numbering system. Each of the four Fourier properties that Lewin names after a familiar collection bears the same number as the unique qualitative genus of which the collection is not only a primary prototype, but also a species in the characteristic ME subgenus. So, for example, the qualitative genus $\mathbb{Q}(12, 3)$ is numbered after Fourier Property 3 (the "augmented-triad property"), and the characteristic ME subgenus of $\mathbb{Q}(12, 3)$ includes the species 3–12 [048] (the "augmented triad").

There is, furthermore, a clear graphic connection between our diagrams of the Fourier balances and Cohn's depiction of the CYCLE homomorphisms (adapted in Figure 1.14), which he describes as "cycles of cycles." This description, in turn, resonates with our generalization of Hanson's projection principle (Figure 2.9) and its attendant invocation of inner and outer cycles and generators. In the interest of getting on with things, we will not give lengthy demonstrations of these connections, but simply state them in terms of our theory of generic prototypes.

Let the notation $\mathbb{F}(c, q)$ mean "Fourier Balance q in the c-pc universe." We will classify $\mathbb{F}(c, q)$ as Class I, II, or III by analogy with its **associated qualitative genus** $\mathbb{Q}(c, q)$ and the latter's characteristic ME subgenus $\mathbb{M}(c, q)$. Each Fourier balance $\mathbb{F}(c, q)$ has $\text{sig}(c, q) = c/\text{gcf}(c, q)$ pans. Each pan contains pitch-classes constituting an exemplar of the ME species $M(c, \text{gcf}(c, q))$, which is always Class I and which we will call its **pan species**. In the event that $\mathbb{F}(c, q)$ is Class I or II — recall that this is always the case in the usual twelve-pc universe — the pan species of $\mathbb{F}(c, q)$ is the smallest species in the characteristic ME subgenus of its associated qualitative genus $\mathbb{Q}(c, q)$; in any event, the pan species is a primary prototype of this genus, and is generated by $\text{sig}(c, q)$. Finally, the transpositional relationship between the pc content of a pan and that of its immediate neighbors is described by $\text{sog}(c, q)$.

Thus we can speak of a Fourier balance having an inner generator, $\text{sig}(c, q)$, that determines within-pan relationships, and an outer generator, $\text{sog}(c, q)$, that determines between-pan relationships. It happens that that Class II Fourier balances have the property that the pc content of each pan is one "semitone" away from its neighbors; this is equivalent to our observation (§ 2.3) that $\text{sog}(c, d) = 1$ for any Class IIa ME species $M(c, d)$. It also happens that the same thing can be said about any $\mathbb{F}(c, 1)$. As a result of the peculiar properties of the number 12, it happens that five out of the six Fourier balances depicted thus far in our investigations have this property. The exception is $\mathbb{F}(12, 5)$, which we discussed in connection with Figure 3.8. In this case, each pan happens to be related to its neighbors by ic 5. The reader is urged to bear in mind that all of this happenstance is just that (rather like the fact that what we call a *fifth* happens to belong to ic 5), and to avoid falling victim to the Intervallic Fallacy by thinking otherwise.

As a foil for discussing some conceptual issues surrounding between-pan relationships, Figure 3.9 shows some Fourier balances from other pitch-class universes that we have already had the opportunity to discuss here and there. With respect to $\mathbb{F}(11, 5)$, we note that the pc in each pan is separated from its neighbors by ic 2 (and not ic 5). This counterintuitive result is a consequence of the counterintuitive definition of the

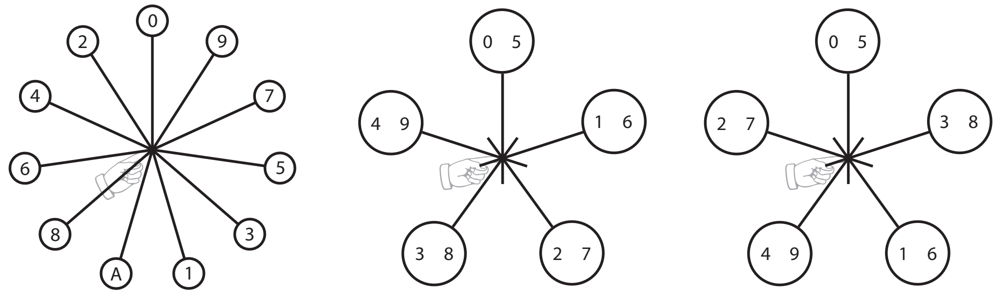


FIGURE 3.9. *Left to right:* $\mathbb{F}(11,5)$, $\mathbb{F}(10,2)$, $\mathbb{F}(10,4)$.

Multiplication Principle, which says — and we are now giving a more general definition than before — that in the Fourier balance $\mathbb{F}(c,q)$, each pc p goes in a pan located at $q \times p$ o'clock (assuming a c-hour clockface). So for $\mathbb{F}(11,5)$, the number 5 means that each pc must be multiplied by 5 to locate its pan: pc 1 is at 5 o'clock, pc 2 is at 10 o'clock, pc 3 is at $15 = 4$ o'clock, and so on. The interval-class separating adjacent pans is not given directly by q, but by $\text{sog}(c,q)$ — *it just so happens* that once in a while $\text{sog}(c,q) = q$, as in the case $c = 12$, $q = 5$.

Figure 3.9 also depicts two Fourier balances from the ten-pc universe. One is the Class II balance $\mathbb{F}(10,2)$, and the other is the Class III balance $\mathbb{F}(10,4)$. These illustrate some general principles that, once again, we will avoid going into in depth, but that are worth mentioning in passing. Every Class III Fourier balance has between-pan relationships that cannot be characterized by ic 1. Furthermore, each is related to a Class II Fourier balance, in the same universe, that has the same pan species; in this case, the pan species of $\mathbb{F}(10,2)$ and $\mathbb{F}(10,4)$ is $[05]_{10}$. In general, the Class III Fourier balance $\mathbb{F}(c,q)$ will have the same pan species as the Class II Fourier balance $\mathbb{F}(c,\text{gcf}(c,q))$, and the former will have an outer generator (describing between-pan transpositions) of $\text{sog}(c,q)$.

§ 3.2 From prototypes to intrageneric affinities

3.2.1 Fuzzification. At this point we are fully prepared to show how our unified theory of generic prototypes connects to the Fourier balances. This is the crux of our entire theoretical program; all of the work on Fourier balances that follows will, thanks to this connection, continue the chain of generalizations (§2.3) from ME chords through the hierarchy of primary, secondary, and tertiary prototypes of the generalized qualitative genus. The connection depends on the fact that the content and arrangement of the pans of a Fourier balance $\mathbb{F}(c, q)$ is determined by the same inner and outer generators that determine the structure of the core prototypes of the qualitative genus $\mathbb{Q}(c, q)$ — the species of its characteristic ME subgenus. We will proceed by describing characteristics of three of the four layers of generic prototypes in terms of the associated Fourier balance and the pan assigned to some arbitrary pc p.

Characteristic ME species: To generate an exemplar of the characteristic ME subgenus, take all of the pcs in the pan containing p, move clockwise, taking all of those pcs, and keep doing this. Stop immediately before reaching the pan that contains $p + 1$. (For Class II species, the procedure will stop after the vary first pan.) To generate the abstract complement of this exemplar, repeat the procedure, but move counterclockwise. The reader may verify that exemplars of the pentatonic and diatonic collections may be generated by carrying out this procedure on $\mathbb{F}(12, 5)$ (Figure 3.8), and that exemplars of the augmented triad and its complement, Messiaen's Mode 3, may be likewise generated on $\mathbb{F}(12, 3)$ (Figure 3.3).

Primary prototypes: To generate an exemplar of a primary prototype, follow the procedure outlined for characteristic ME species, but stop at any point whatsoever after taking all of the pcs in a pan. The reader may verify that exemplars of 4–9 [0167], its complement, and 6–7

[012678] may be generated by carrying out this procedure on $\mathbb{F}(12, 2)$ (Figure 3.6).

Tertiary prototypes: To generate an exemplar of a tertiary prototype, take arbitrary pcs one at a time from any pan; once a pan has been exhausted (and only then), proceed to the next pan clockwise and continue. Stop at any point. The procedure may also be carried out counterclockwise. The reader may verify that exemplars of the Petrushka chord 6–30 [013679] may be generated by carrying out this procedure on Fourier Balance $\mathbb{F}(12, 4)$ (Figure 3.3).

These procedures describe necessary and sufficient conditions on the various kinds of prototypes: every prototype can be generated with them, and nothing other than a prototype can be generated with them. Indeed, learning and internalizing these procedures may make it considerably easier to understand clearly the theory-unifying concepts couched in the occasionally stultifying (but necessary) mathematical language of Chapter 2.

The foregoing discussion can be summarized by observing that a prototype of the qualitative genus $\mathbb{Q}(c, q)$ has pcs clustered together as closely as possible on adjacent pans of the Fourier balance $\mathbb{F}(c, q)$. In relation to our interpretation of each of Lewin's Fourier properties as *balance* on some Fourier balance, we note that generic prototypicality may be interpreted as maximal *imbalance* on the associated Fourier balance — at least to the extent that *a generic prototype tips its associated Fourier balance more than any other chord of the same cardinality possibly can.* The italicized words in the previous sentence constitute a single, unified, and conceptually simple condition that is both necessary and sufficient for prototypicality as it was defined with substantially more theoretical overhead in Chapter 2. We are about to develop this point further, but before descending into technicalities again we underscore the importance of interpreting generic prototypicality as maximal Fourier-balance imbalance: under this interpretation, prototypicality has become the limit case of a phenomenon that comes in degrees. In other words, we have fuzzified the concept of a prototype.

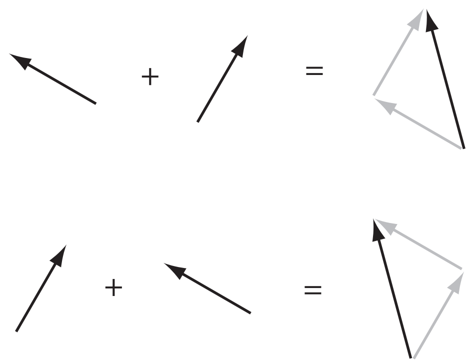


FIGURE 3.10. *The principle of arrow addition, which is commutative.*

That said, we will descend into the aforementioned technicalities. Our metaphorical Fourier balances tip when the force of a pitch-class is applied to them. They tip maximally when the forces are applied to what can be loosely characterized as "unbroken spans" of adjacent pans of the balance, and less so when the spans are "broken." Buchler's treatment of broken and unbroken interval cycles (2000) uses this language too, but in the present context of inner and outer cycles, the terms lose their grip somewhat; the metaphor of Fourier balances allows us to circumvent the slippage by appealing directly to the notion of force. In elementary physics, forces are represented as arrows that can be added to one another in the manner depicted in Figure 3.10 — to add two arrows, move them relative to each other (taking care not to rotate them) so that the tail of one is attached to the head of another; then draw a new arrow from the free tail to the free head. The new arrow is the sum of the original two. Figure 3.10 also shows that arrow addition is commutative, meaning that the order in which arrows are added does not affect their sum. When the arrows being added have the same length and point in opposite directions, they sum to what we might think of as a "zero arrow" (see Figure 3.11) — under the interpretation of arrows as forces, this corresponds to equilibrium, a situation in which forces annihilate each other.

On Fourier Balance 1, each pitch-class *n* tips the balance toward *n* o'clock. To model the force it exerts on the balance, then, we can use an arrow of unit length oriented to

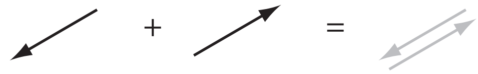


FIGURE 3.11. *Two mutually inverse arrows sum to a zero arrow.*

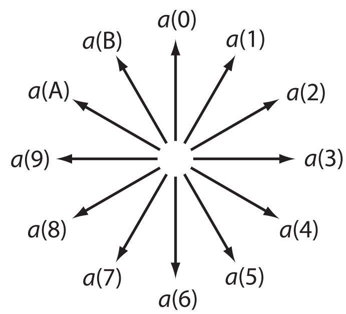


FIGURE 3.12. *Association of arrows with Fourier Balance 1.*

point toward n o'clock. We will name such arrows $a(n)$, and refer to the unit of force or length that they represent as a **lewin**, abbreviated **Lw**. Figure 3.12 displays the gamut of one-lewin arrows $a(n)$, each of which corresponds to the force exerted by the pitch-class n on Fourier Balance 1.

Fourier Property 1 is the property of being balanced on Fourier Balance 1, which in turn is the property of comprising pitch-classes $p_0, p_1, \ldots, p_k$ such that the arrows $a(p_0), a(p_1), \ldots, a(p_k)$ form a closed loop when arranged head-to-tail, thus summing to the zero arrow. The smallest closed loop is the one with two "sides," which is the case depicted in Figure 3.11 and is the configuration of any annihilating pair (tritones on Fourier Balance 1). The next smallest is the one with three sides in the form of an equilateral triangle; all annihilating triples (augmented triads on Fourier Balance 1) are of this type. All larger loops can be decomposed into annihilating pairs and triples — Figure 3.13 illustrates the decomposition of the chord {123679A}, which has Fourier Property 1, into two annihilating pairs (α and γ) and an annihilating triple (β).

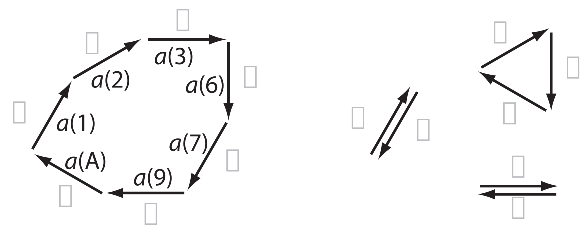


FIGURE 3.13. *Decomposition of a chord with Fourier Property 1 into arrow cycles.*

By way of the Multiplication Principle, the foregoing characterization of Fourier Property 1 generalizes as follows: Fourier Property n is the property of being balanced on Fourier Balance n, which in turn is the property of comprising pitch-classes $p_0, p_1, \ldots, p_k$ such that the arrows $a(np_0), a(np_1), \ldots, a(np_k)$ form a closed loop.

So much for balance; the quantification of *imbalance* — which will cause a theory of generic affinity to emerge fully-formed from our theory of prototypes — is enabled by the technique of arrow addition. In particular, each Fourier balance represents a particular way of associating pcs with arrows, and the degree to which a chord is imbalanced on that Fourier balance (its intrageneric affinity, or degree of prototypicality) is proportional to the length of the arrow resulting from the addition of the arrows associated with the constituent pcs of that chord. When a chord is balanced on the Fourier balance, its pc-arrows sum to the null arrow (they annihilate each other); but when a chord is not balanced, its pc-arrows reinforce each other and sum to a long arrow.

Consider Fourier Balance 1. Each pitch-class exerts the same amount of force on the balance (1 Lw), but in a unique direction. Because of the way arrow addition works, the imbalance caused by a chord on Fourier Balance 1 is tied to the degree to which the forces of its constituent pcs tend to point in the same direction. Clearly the way to maximize this imbalance, given the structure of Fourier Balance 1, is to select pcs clustered around a single point in pc-space — more or less the defining characteristic of $\mathbb{Q}(12, 1)$-prototypicality. Under such a circumstance, the arrows will tend to reinforce one another. Figure 3.14, with respect to which Buchler's concept of broken cycles (2000) is quite suggestive, works out three examples of pentachords that exert a great

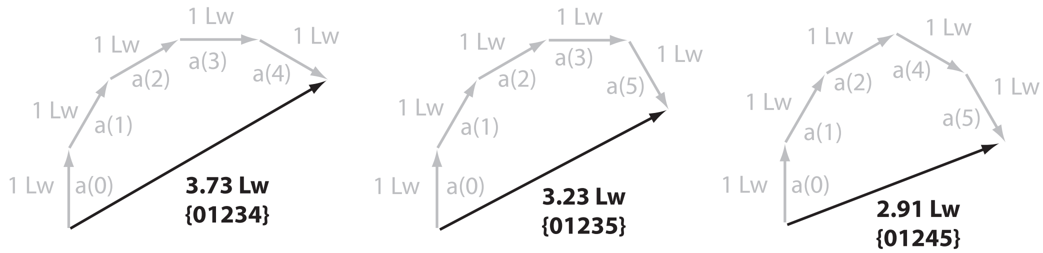


FIGURE 3.14. $\mathbb{F}_{12,1}$ *for highly prototypical pentachords.*

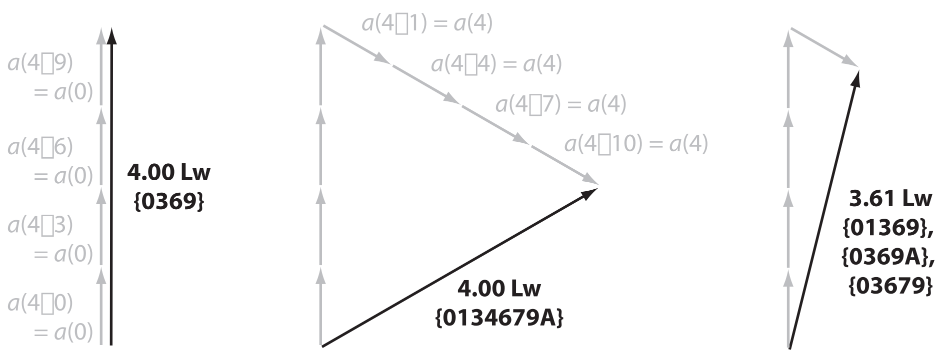


FIGURE 3.15. $\mathbb{F}_{12,4}$ *for Messiaen's prototypes.*

deal of tipping force on Fourier Balance 1. Figure 3.15 demonstrates some prototypes of $\mathbb{Q}(12, 4)$ — specifically, the ones we discussed in connection with the Messiaen excerpt at the end of Chapter 2 — on its own Fourier balance.

3.2.2 Theoretical fallout. Transposing, inverting, or complementing a chord does not affect the amount of tipping force it exerts on Fourier Balance 1, only the direction. This is relatively easy to grasp in the case of transposition or inversion, since in such cases the individual chord arrows undergo some rotation or reflection that is carried over to any sum-arrow corresponding to a chord. Complementation always effects a reversal of the arrow, which is the same as a half-rotation. This somewhat surprising result is easily explained: (1) the aggregate has Fourier Property 1, since it can be partitioned into annihilating subsets; (2) any chord added to its complement forms the aggregate; (3) the force exerted by a chord on Fourier Balance 1 annihilates the force exerted by its complement.

The foregoing observations apply equally, in fact, to every Fourier balance, and will allow us to consolidate our understanding of exactly how far we have come. The degree of prototypicality of any chord with respect to any qualitative genus is invariant under transposition and inversion (and complementation), which brings us back to the assertion, made at the very outset, that generic prototypicality is a property that supervenes on — abstracts certain essential aspects or facets of — its pcset-class (species) identity. The idea that these are essential aspects and not accidental aspects, like we have been claiming interval content to be, is supported by an overwhelming argument from theoretical unification. This argument would be empty, however, were it not for the fact that they reflect so many of the accidental properties modeled by ad-hoc approaches. Such ad-hoc approaches are motivated by intuitions, and while I have consistently argued that intuitions make flimsy theoretical foundations by themselves, no purely nominalist theory of musical entities — however solidly constructed — can come alive with meaning until it makes a connection with intuition.

So what it does it "mean" for a chord to exert a large amount of tipping force on Fourier Balance 1 — to be highly prototypical of the genus $\mathbb{Q}(12, 1)$? It should be quite clear by now what the essential *nominal* meaning is, but there are a number of different ad-hoc ways to *intuitively* characterize this meaning in the standard language of pcset theory; some examples follow.

- Consider the average span of each dyad contained in p, where **span** is to be construed as the interval-class number associated with that dyad — this is quite closely related to Block and Douthett's (1994) approach to measuring "evenness." The more $\mathbb{Q}(12, 1)$-prototypical p is, the lower this average.

- More generally, consider the span of any chord to be the highest number (*qua* pitch-class name) of the prime form of its species. Consider the average span of p and all of its subsets (down to dyads). The more $\mathbb{Q}(12, 1)$-prototypical p is, the lower this average.

- Choose your favorite fuzzy similarity relation, then consider the species corresponding to a contiguous chunk of the chromatic scale of the same cardinality of p — that is, the species labeled k–1 on Forte's roster, where k is the cardinality in question. Consider the measured similarity of p to this "chromatic" species. The more $\mathbb{Q}(12, 1)$-prototypical p is, the greater this similarity.

The table in Figure 3.16 displays the $\mathbb{Q}(12, 1)$-prototypicality (in lewins) of each tetrachord-class and collates it with the three approximations just described, using Castrén's RECREL (1994), which produces lower numbers to indicate greater similarity, as the similarity relation. Clearly all of these measures correlate with one another; the argument from theoretical unification suggests a causal role for generic prototypicality — our model of which is, quite indisputably, far from ad hoc.

When concept of intrageneric affinities first arose (§ 1.2.2) it was suggested that they might be modeled by using similarity relations to measure the closeness between an arbitrary chord and a generic prototype, as we have just done with RECREL. This raised the obvious issue of selecting a similarity relation to do the job — a truly difficult proposition given the market glut. As it has turned out, however, our fuzzification of prototypicality has obviated the need for using any similarity relation at all, at least in this context.

One suggestive difference between measuring prototypicality by imbalance and measuring it with a similarity relation is that we are able to measure intrageneric affinity — distance from a prototype — in meaningful units. But since the unit of measurement here, 1 Lw, is defined as the unit of tipping force exerted by a single pitch-class on a Fourier balance, there is a sense in which a tetrachord (since it has four pitch-classes) should be able to tip any Fourier balance by 4 Lw. As it happens, just one type of

	$\mathbb{F}_{12,1}$ (Lw)	mean ic	mean span	RECREL
4–1 [0123]	3.346	1.67	2.09	0.00
4–2 [0124]	2.909	2.17	2.73	0.23
4–3 [0134]	2.732	2.33	2.91	0.33
4–10 [0235]	2.449	2.67	3.36	0.33
4–4 [0125]	2.394	2.67	3.36	0.36
4–11 [0135]	2.236	2.83	3.55	0.40
4–7 [0145]	1.932	3.00	3.73	0.54
4–5 [0126]	1.932	3.17	4.00	0.50
4–12 [0236]	1.932	3.17	4.00	0.51
4–13 [0136]	1.732	3.33	4.18	0.51
4–21 [0246]	1.732	3.33	4.18	0.67
4–6 [0127]	1.732	3.33	4.36	0.50
4–14 [0237]	1.506	3.33	4.45	0.53
4–17 [0347]	1.414	3.33	4.45	0.67
4–Z15 [0146]	1.414	3.50	4.36	0.54
4–Z29 [0137]	1.414	3.50	4.45	0.53
4–22 [0247]	1.239	3.50	4.64	0.58
4–19 [0148]	1.000	3.50	4.81	0.71
4–8 [0156]	1.000	3.67	4.55	0.67
4–18 [0147]	1.000	3.67	4.73	0.69
4–24 [0248]	1.000	3.67	4.91	0.71
4–23 [0257]	0.897	3.67	4.82	0.54
4–26 [0358]	0.732	3.67	4.91	0.67
4–20 [0158]	0.518	3.67	4.91	0.71
4–16 [0157]	0.518	3.83	4.91	0.67
4–27 [0258]	0.518	3.83	5.00	0.69
4–9 [0167]	0.000	4.00	5.00	0.67
4–25 [0268]	0.000	4.00	5.09	0.67
4–28[0369]	0.000	4.00	5.55	0.83

FIGURE 3.16. *Measuring and approximating* $\mathbb{F}_{12,1}$ *for tetrachord types.*

tetrachord, 4–28 [0369], can tip just one Fourier balance, $\mathbb{F}(12, 4)$, that much. The maximal tetrachordal imbalance on the other balances is less; according to Figure 3.16, for instance, the prototypical tetrachords exert 3.346 Lw of force on Fourier Balance 1. The reason for this discrepancy is that in order for a chord of cardinality d to tip a Fourier balance by d Lw, all of the arrows must point in *exactly* the same direction. This is only possible if the chord is a subset of, or equal to, some exemplar of the Fourier balance's pan species. Otherwise, the best the chord can do is to have pcs gathered in adjacent pans, so that the arrows point in *approximately* the same direction. In a sense, this is the very definition of prototypicality expounded at the beginning of this section.

We have been observing that the maximal imbalance a chord can exert on a Fourier balance is constrained to a certain extent by the balance's structure; it is constrained to a much greater extent by the chord's cardinality. No chord of cardinality d can exert an imbalance greater than d Lw. Does this mean that with respect to, for example, $\mathbb{Q}(12, 2)$, the prototype 3–5 [016] is less prototypical, at 2.65 Lw, than the non-prototype 4–8 [0156], at 3.00 Lw? One may wish to answer the question either way. From one point of view, the trichord ought to get a better "score" than the tetrachord, in which case one might wish to scale the prototypicality value with reference to the potential maximum (2.65 Lw for trichords, 3.46 Lw for tetrachords), in which case the trichord comes out ahead. From another point of view, we might wish simply to find an interpretation of the given data — perhaps taking a probabilistic approach, as Lewin was fond of doing (e.g., 1977; 1979), and saying that the trichord is deservedly handicapped by virtue of having fewer "competitors" of the same cardinality than the tetrachord, which has risen to near-prototypicality within a much larger field. In a sense, however, these considerations are somewhat silly, since they are taking place outside of any theoretical or analytical context that would enliven them. Strictly within the context of our theory, the numbers are different simply because 3–5 exerts less force on $\mathbb{F}(12, 2)$ than 4–8; what that *means* is up for grabs.

A final argument against the Intervallic Fallacy, which doubles as a final demonstration of theoretical unification, will conclude our theoretical study of affinities. The table in Figure 3.17 lists the hexachordal species with the two highest levels of ic-4 content; all have either four or six instances of ic 4. The two hexachords with six instances of ic-4 represent opposite poles of a continuum — on one end is 6–20 [014589], the hexatonic collection, a secondary prototype of $\mathbb{Q}(12, 3)$ with Lewin's whole-tone-scale property; and on the other is 6–35 [02468A], the whole-tone scale, a primary prototype of $\mathbb{Q}(12, 6)$ with the augmented-triad property. Thus each is a prototype of one genus, and an "antiprototype" of the other. The nine hexachords with four instances of ic-4 sit, in three groups of three, at various points along this hexatonic–whole tone continuum

How this constitutes evidence against the Intervallic Fallacy should be clear — all of these chords are relatively saturated with ic 4, yet they span two genera rather evenly. This is the very problem that, as we saw in Chapter 1, led Hanson (1960), Eriksson (1986), and Buchler (2001), among others, to develop ad-hoc workarounds to avoid. Hanson and Buchler's workarounds, however, which more or less amount to the same

	$\mathbb{F}_{12,3}$	$\mathbb{F}_{12,6}$	ic 4	ic 1	ic 3	ic 5	ic 2	ic 6	4–19	5–21	[048]
6–20 [014589]	4.24 Lw	0.00 Lw	6	3	3	3	0	0	6	6	[048]
6–14 [013458]	3.16	0.00	4	3	3	3	2	0	3	1	[024]
6–Z19 [013478]	3.16	0.00	4	3	3	3	1	1	3	1	[026]
6–Z44 [012569]	3.16	0.00	4	3	3	3	1	1	3	1	[026]
6–15 [012458]	2.83	2.00	4	3	3	2	2	1	2	1	[014]
6–16 [014568]	2.83	2.00	4	3	2	3	2	1	2	1	[015]
6–31 [014579]	2.83	2.00	4	2	3	3	2	1	2	1	[037]
6–21 [023468]	1.41	4.00	4	2	2	1	4	2	1	0	[014]
6–22 [012468]	1.41	4.00	4	2	1	2	4	2	1	0	[015]
6–34 [013579]	1.41	4.00	4	1	2	2	4	2	1	0	[037]
6–35 [02468A]	0.00	6.00	6	0	0	0	6	3	0	0	[048]

FIGURE 3.17. *Hexachord species with high ic-4 content.*

thing, apply only to the prototypes of their systems; only Eriksson's solution (ic-vector models) has more general applicability. Indeed, we can see in Figure 3.17, as we proceed from one end of the continuum to the other, a general transformation from Eriksson's model IV (high ic-4 content; some odd-ic content; low ic-2 and -6 content) to model II (high even-ic content; low odd-ic content).

More interesting than the evidence against the Intervallic Fallacy, though, are the new questions these data raise. We have noticed that the nine non-prototypical species are gathered in three groups of three. Each group contains one M-invariant species, and the other two species are M-transforms of one another. Each group is a set-group in Morris's (1982) $SG(\alpha)$, $SG(\beta)$, and $SG(\delta_2)$. Each group can be characterized entirely by the number of times it embeds the species 4–19 [0148], the unique tetrachordal secondary prototype of $\mathbb{Q}(12, 3)$, and is consistent with respect to the number of times it embeds 5–21 [01458], the unique pentachordal secondary prototype of the same genus. All of these observations seem somehow interrelated, and suggest that further study of what we called *algebraic genera* in Chapter 1 is warranted.

The last column of Figure 3.17 shows what remains if an augmented triad is removed from an exemplar in each species. Most obviously in the case of the two triples "closest" to the whole-tone scale, relationships among these residual trichords suggest some ways of making connections between the triples. Figure 3.18 shows some more, detailing the relationships under M as well as connections via Forte's (1973) similarity relations R_1, R_2, and R_p (the latter of which can be interpreted as pertaining to voice-leading).

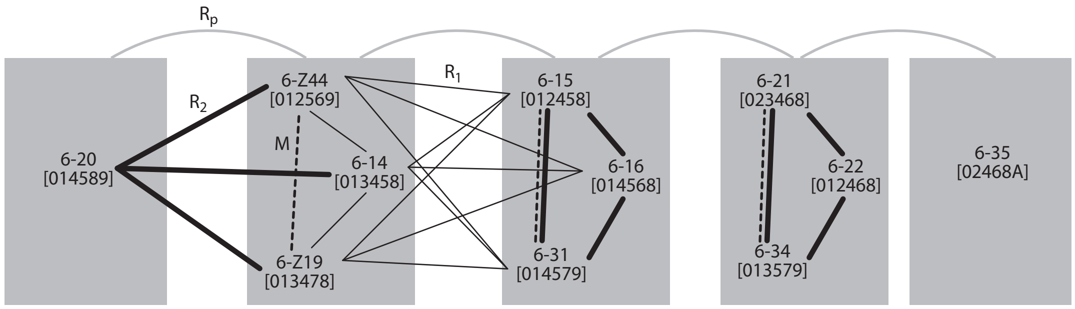


FIGURE 3.18. *Additional relationships among the species in Figure 3.17.*

Many things could be said about Klumpenhouwer-network interpretations of chords in these species, and about fuzzy similarity relations among them. This rapid-fire list of observations could be continued for quite awhile, but the point is clear: while there is a certain amount of intellectual satisfaction that derives from the unification of theories, that satisfaction cannot compare to the dazzling array of new research directions that appear in its wake.

The work we have accomplished so far in this chapter generalizes our theory of prototypes by interpreting it as a special case of a more general theory. Lewin's theory of Fourier properties is still another special case, one which pertains to a voice-leading problem. The more general theory seems to admit of an interpretation as a theory of affinities for our prototypes: first, our working metaphor entails that prototypicality is the limit case of Fourier-balance imbalance, a phenomenon that comes in degrees; second, the Multiplication Principle shows exactly how the Fourier balances, and therefore their associated qualitative genera, are related to each other. The former gives us a way to reckon intrageneric affinities. The latter neatly and completely explains intergeneric affinities, about which we will have nothing further to say in this study, except to note (a) that the Multiplication Principle is nothing but another way of describing what we might have called the Division Principle — the concept, which was central to the development of the previous chapter's prototype theory, that (maximally) even divisions of the pitch-class universe structure the entire taxonomy of chords — and (b) that the Multiplication Principle clearly resonates with our discussion, in Chapter 1,

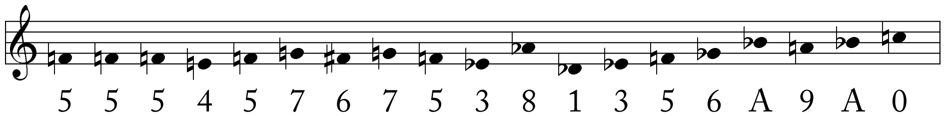


FIGURE 3.19. *Ligeti,* Lux aeterna*, beginning of first microcanon*

of approaches to generalizing the M_5 and M_7 operators suggested by Morris (1982), Cohn (1991), and Harrison (2000).

§ 3.3 Notes on Q-space.

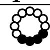
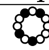
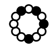
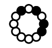
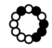
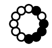
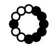
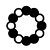
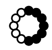
We will now pick up the notion, mentioned back in Chapter 1, that qualitative relationships among chords have a spatial character, taking as our point of departure the opening of Ligeti's *Lux aeterna*. Much of the piece is structured by what Clendinning (1995) calls "microcanon," a technique involving the simultaneous presentation of some melody in a relatively large number of voices, with each voice proceeding through the melody at an independent and flexible pace. The musical result is a blurring of the distinction between the vertical and the horizontal as melodic successions become harmonic simultaneities.

Figure 3.19 shows the beginning of the pitch succession whose microcanonic setting opens the piece. In a detailed analysis, Bernard (1994) notes the tendency of the melody to fill chromatic space: "the edges of occupied space creep outward here even though the tone and semitone progressions are not always direct successions" (p. 231). Bernard is speaking of pitch space here, although our emphasis has been on pitch-class space; in the present case, however, both spatial metaphors apply. Early on, the entry of the G "overshoots" the creeping expansion, but the immediate advent of F♯ fills in the gap, suggesting that the space-filling process is normative even when it does not happen in a tidy fashion (indeed, Ligeti tends to avoid tidiness in his music.) The initial entrance of the B♭ works similarly, opening up a space that is filled in by A♮ a moment later.

By "occupied space" Bernard refers to space that is filled in a more or less compact fashion; chords that fill space in this way exert great force on $\mathbb{F}(12, 1)$. At the appearance of G, the total harmonic state corresponds to the chord $\{457\}$, which is characteristic, but not prototypical, of $\mathbb{Q}(12, 1)$; the arrival of F♯ yields the prototypical $\{4567\}$. Later in the process, the lowest and highest notes of the melody (D♭4 and C♮5 respectively) are involved in the two real violations of the space-filling principle; any expectation these pitches create for the introduction of D♮4 or B4 remains unfulfilled for the duration of the canon (which continues well beyond the pitches shown in Figure 3.19). Bernard calls special attention to the symmetry of the almost-filled D♭4—C5 space, whose two "missing" pitches are inversions of each other about F♯4 and G4. This structure fills a sufficiently broad span of pitch space that to treat it as a chord of pitch-classes would make little musical sense. What is worth noting, however, is that the "intuitive" idea of filling harmonic space, central to Bernard's analysis, is precisely what an imbalanced Fourier balance is good at reflecting, particularly in cases where that space-filling is imperfect. It is worth mentioning that our approach to chord quality is not conceptually bound to the notion of pitch-class space, octave-equivalent or otherwise; qualitative categories in pitch space are simply those $\mathbb{Q}(c, q)$ where $c = \infty$, although admittedly calculus becomes necessary in that case.

Now that our foot is in the door, let us examine some of the actual simultaneities that arise in the working-out of the first microcanon. Since Ligeti's technique involves a certain amount of controlled chaos, especially in the rhythmic domain, it would be beside the point to make a complete catalogue of all such simultaneities. Instead, Figure 3.20 shows a "random" sample of harmonic states taken at the downbeats of bars 6–14. (In the not infrequent case that a voice is resting at a downbeat, the last pitch sung by that voice is presumed to continue contributing to the harmonic state at the downbeat.)

The excerpt under consideration begins at the point that four distinct pitches are heard and continues until both of the boundary pitches of the opening microcanon (D♭4 and C5) are heard. It spans seven chords numbered $p_0, p_1, \ldots p_7$, which form a harmonic progression from $\{4567\}$, a chord prototypical of $\mathbb{Q}(12, 1)$, to the five-flat diatonic collection, a prototype of $\mathbb{Q}(12, 5)$. The intervening chords seem to make a gradual transition between the two prototypes, a transition we might characterize

n	p_n	$\mathbb{F}_{12,1}(p_n)$	$\mathrm{M}_5(p_n)$	$\mathbb{F}_{12,5}(p_n)$
1		3.35 Lw		0.90 Lw
2		2.91 Lw		1.24 Lw
3		2.24 Lw		2.24 Lw
4		1.51 Lw		2.39 Lw
5		1.88 Lw		3.26 Lw
6		1.24 Lw		2.91 Lw
7		0.27 Lw		3.73 Lw

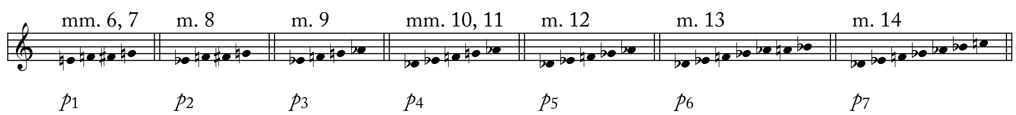


FIGURE 3.20. *Ligeti,* Lux aeterna *(opening): harmonic states at downbeats.*

as moving from a chromatic quality to a diatonic quality. This gradual transition is reflected in the table in two ways: first, the force applied to the Fourier balance $\mathbb{F}(12, 1)$ *decreases* more or less steadily as the progression continues; and second, the force applied to $\mathbb{F}(12, 5)$ *increases* more or less steadily at the same time. The coincidence of these two features raises the question of whether they are causally related in some way, whether the "diatonic" and "chromatic" genera, $\mathbb{Q}(12, 1)$ and $\mathbb{Q}(12, 5)$, are categorically opposite to one another such that an increase in affinity to one necessitates a decrease in affinity to the other — this is an assumption made by (Vieru, 1993), among many others. This question does not admit of an unequivocal answer, because its premises fail to address fully the complexity of the relationship between these two genera; we will now do so.

Figure 3.21 displays a Cartesian plane whose vertical and horizontal axes represent $\mathbb{Q}(12, 1)$- and $\mathbb{Q}(12, 5)$-affinity, interpretable as "chromaticness" and "diatonicness," respectively, and graphs the seven chords of Figure 3.20 on that plane. It is immediately obvious that these seven points do not lie along a straight line, exponential curve, or any other tidy structure, and we may therefore infer that the two genera are not fixed in straightforward opposition to one another. It is not the case, however, that Fourier-balance forces for these two genera are entirely independent of one another either. The

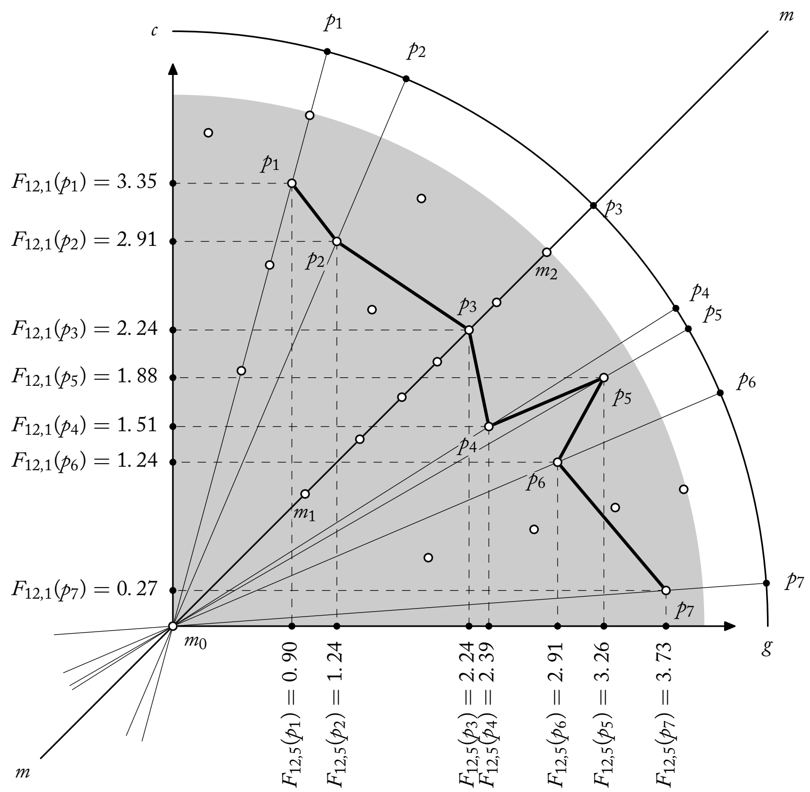


FIGURE 3.21. *Modeling a "chromatic-to-diatonic" transition from the appropriate viewpoint.*

unlabeled white dots in Figure 3.21 represent places where other chords would lie under the same graphing technique; every possible chord lies at one of these 24 places. The shaded portion of the graph emphasizes the fact that all 24 points lie within an area bounded by a quarter-circle whose radius is 4 Lw, which allows us to construct the following inequality relating the force an arbitrary chord exerts on the Fourier balances in question:

$$\mathbb{F}_{12,1}(x)^2 + \mathbb{F}_{12,5}(x)^2 < 16.$$

Imagine, for example, that a chord x exerts 3 Lw of force on $\mathbb{F}(12, 1)$. Then $\mathbb{F}_{12,5}(x)^2 < 16 - 9$ Lw, so it must exert less than $\sqrt{7} \approx 2.65$ Lw of force on $\mathbb{F}(12, 5)$. We may therefore state with some confidence that chromaticness and diatonicness, while in some sense independent of one another, do *constrain* one another to the extent that no chord can be prototypical of both.

The last sentence, of course, is not entirely true — a singleton is indeed prototypical of both genera, in the technical sense that a singleton is prototypical of every genus. This raises the issue of whether prototypicality is absolute or relative to cardinality, an issue we dismissed earlier in the absence of a context that would give it meaning. In the present context, we are assuming a certain interpretation of the genera (as modeling the qualitative predicates *chromatic* and *diatonic*); we are discussing an aspect of a particular chord progression that seems to be independent of the cardinalities of the chords involved; and we are also dealing with chords that lie close to the limit of the space of possibilities shaded gray on the graph. In connection with these issues, and with the issue of the interdependence of the two genera under discussion, we note that it is quite possible to find chords that are not prototypical (in the absolute, cardinality-independent sense) of either genus. Any primary prototype of any other genus, for instance, will have Lewin's exceptional property, which is to say that it will exert no force whatsoever on either $\mathbb{F}(12,1)$ or $\mathbb{F}(12,5)$. Secondary and tertiary prototypes of the other genera can be expected to exert only a small amount of force on these balances. We may conclude that the evident inverse relationship of qualitative forces in the Ligeti passage is something special to that group of chords, since none of those chords acts weakly on both Fourier balances at once.

The chord p_3 from the Ligeti excerpt is M_5-invariant up to transposition, as the relevant clockface diagrams in Figure 3.20 make graphically clear. The Multiplication Principle guarantees that all such chords will exert an equal amount of force on $\mathbb{F}(12,1)$ and $\mathbb{F}(12,5)$. In Figure 3.21, a diagonal line whose endpoints are labeled m describes all points equidistant from both axes. All M_5-invariant species, including p_3, will lie along this line when graphed on the plane, at one of the eight points indicated. Their distance from the origin tells us something significant; chords lying distant from the origin, such as Ligeti's p_3, already mentioned, and exemplars of 6–8 [023457] — the only all-combinatorial hexachord species that is not a generic prototype, and whose exemplars lie at the extreme point m_2 — have characteristics of both $\mathbb{Q}(12,1)$ and $\mathbb{Q}(12,5)$. On the other hand, species that lie close to the origin, such as the primary and purely secondary prototypes of $\mathbb{Q}(12,4)$, which lie at the points labeled m_0 and m_1, respectively, have little in common with $\mathbb{Q}(12,1)$ and $\mathbb{Q}(12,5)$.

We can broaden the claims just made by taking advantage of an alternative way of describing points in a plane. The two **polar coordinates** of a point are, first, its distance from the origin, and second, its angular orientation relative to the axes. The arc with endpoints labeled c and g in Figure 3.21 will help to visualize this, as the seven points corresponding to successive harmonic states in the Ligeti excerpt have been projected onto the arc. Their projected position is determined entirely by their angular orientation, and our initial observation about the "gradual transition between the two prototypes" is reflected in the fact that the harmonic states' angular orientation with respect to the orthogonal axes does indeed progress monotonically from an orientation toward the $\mathbb{F}_{12,1}$ axis to an orientation toward the $\mathbb{F}_{12,5}$ axis. This is of particular analytic interest in light of the fact that p_5 comes "out of order" from a strictly $\mathbb{Q}(12,1)$-oriented or strictly $\mathbb{Q}(12,5)$-oriented point of view, as can be seen either by examining the relevant columns in Figure 3.20 or by noting the kink at p_5 in Figure 3.21.

Two methodological points follow from the constrained independence of these two genera. First, we have found it necessary to take them both into account in describing the "gradual transition" under scrutiny; attention to the angular position of the graphed harmonic states synthesizes the $\mathbb{Q}(12,1)$-oriented and $\mathbb{Q}(12,5)$-oriented points of view. In order to model the idea of the chromatic-diatonic continuum by means of intrageneric affinities, we have taken a particular viewpoint on the spatial relationships those affinities imply. Second, we must recognize that while the mutual constraints that the two genera place on each other enables a musical situation in which they are opposed to one another, it need not always be the case — that is, the viewpoint we have taken is not the only one possible. Figure 3.22 amplifies these points by graphing the harmonic states leading up to the beginning of the gradual transition we have been considering (recall Figure 3.19 and the associated discussion). The piece begins with a unison F, which is joined first by the E below and then by the G above; the gap in chromatic space is then filled in by F♯, at which point the gradual transition to a diatonic collection begins. The harmonic

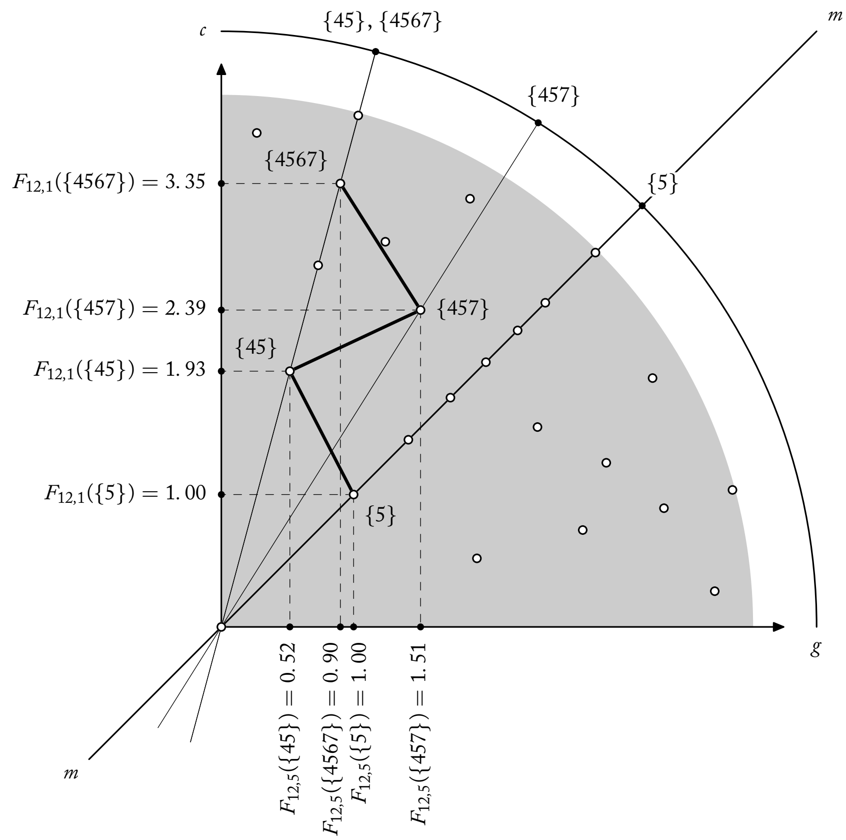


FIGURE 3.22. *Modeling increasing "chromaticness" necessitates a different viewpoint.*

states of this buildup are described by the sequence of chords {5} — {45} — {457} — {4567}. As Figure 3.22 makes plain, these chords exert increasing force on $\mathbb{F}(12, 1)$; yet their action on the other Fourier balance does not follow any similarly teleological progression. An attempt to situate these four chords on a continuum by observing their angular orientation fails to produce interesting results. In this case, our "intuitions" about space-filling have a distinct bias toward $\mathbb{Q}(12, 1)$, and require a viewpoint that reflects that bias.

Figure 3.23 summarizes the previous two figures and our discussion of the passage's harmonic character. Our discussion, like the theory in which it is grounded, has been complex. But the observations are natural and intuitive: we start by gradually filling in chromatic space until a chromatic tetrachord is achieved, then the chord gradually transforms into a diatonic collection. At the point where the chromatic tetrachord is achieved, the progression turns a corner; it is only at that point that it becomes

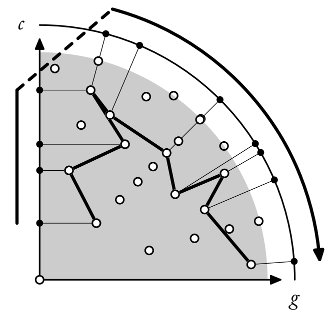


FIGURE 3.23. *One progression, two viewpoints.*

necessary or useful to think about an opposition between *chromatic* and *diatonic* as qualitative categories. (Indeed, to think about that opposition too early makes it more difficult to assert the process from the opening singleton to the chromatic tetrachord.) By "the progression turns a corner," of course, we mean that our viewpoint — our geometric interpretation of the affinity data — turns a corner.

This analytical example has been simple, but through its simplicity it clearly demonstrates the nuanced sort of qualitative claims that are easy to make using commonplace language. The theory we have developed is capable of modeling such claims, but much more importantly, it also grounds them in a solid and powerful conceptual foundation. The spatial methodology we have been using generalizes quite easily, and suggests that the "Q-space" discussed in passing in Chapter 1 can be described with the same technique — interpreting intrageneric affinities, measured by Fourier-balance forces, as coordinates in a multidimensional space. We have been operating so far in a two-dimensional projection of that space, but the analytical possibilities of Q-space as a whole are unlimited. The space promises to close the book on the proliferation of similarity relations, at least the qualitative ones, for good. A theory of distances in Q-space, measured in meaningful units and describing relationships among chords whose spatial distribution is itself meaningful, will provide everything similarity relations have been trying to accomplish; the larger theoretical context will provide even more.

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