OF THE DIFFERENT GENERATORS OF A SCALE

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Foreword

In $\mathbb{Z}_c$, taken as a model of an equal temperament chromatic universe with c pitch classes, many musical scales can be constructed by the iteration of some generating interval f, usually chosen as coprime with the chromatic cardinality c. Of course one gets the same scale (starting from the other end) if one uses as generator the opposite interval $-f = c - f$. The most famous examples in 12 tone universe are the major scale and the pentatonic, which are generated by fifths ($f = 7$) or fourths ($-f = 5$): (5 0 7 2 9 4 11) or (11 4 9 2 7 0 5) are two ways of enumerating the pcs in C major.

This short paper answers the following simple question: what are all the possible generators of a given scale ?

This seems too obvious to be worth considering: an arithmetic sequence of numbers has clearly only two (opposite) generators, that are computed as the difference between two consecutive terms. So has a generated scale exactly two generators, too ?

The answer is not so obvious, however, because in a cyclic group, such as the chromatic universe $\mathbb{Z}_{12}$ or in general $\mathbb{Z}_c$, an arithmetic sequence coils around itself. For instance, the 'almost full' scale, with $c-1$ notes, is generated by any interval f coprime with c: the multiples of f mod c, being $c-1$ different pc's, build up such a scale[1]. This yields in general much more than two generators. With another extreme case, the 'one note scale', any number can be viewed as a generator as the scale is reduced to its starting note. The same could be said in the metaphysical cases of empty or full aggregate scales. Is it still so 'obvious', now, that a $c-2$ or $c-3$ note scale has no more than two generators ?

Another frequent case with a wealth of different generators is when f is not coprime with c, which happens when the cardinality of the scale is not coprime with c: then $k \times f$ generates the same scale as f (up to transposition), for any k coprime[2] with c, that is to say the generators are all elements whose order in the group $\mathbb{Z}_c$ is equal to some particular divisor of c. Consider for instance a whole-tone scale in 14 tone temperament, which exhibits 6 generators:

$$(0\ 2\ 4\ 6\ 8\ 10\ 12) = (0\ 4\ 8\ 12\ 2\ 6\ 10) = (0\ 6\ 12\ 4\ 10\ 2\ 8) \text{ and their reverses.}$$

Mathematically speaking, a subgroup of $\mathbb{Z}_c$ with d elements is generated by precisely $\Phi(d)$ intervals, Φ being Euler's totient function. Conversely, a subgroup of $\mathbb{Z}_c$ generated by f is the (one and only) cyclic subgroup with $c/\gcd(c, f))$ elements. One last example: the 'incomplete whole-note scale' in ten notes temperament (1 3 5 7) has 4 different generators (with as many different starting points).

For a final vindication of this question, let us observe that other modes of generating pc-sets in $\mathbb{Z}_c$, maybe less familiar, exhibit a similar profligacy of generators. For instance the *powers* of 3, 11, 19 or 27 modulo 32 generate the same 8-note scale in $\mathbb{Z}_{32}$, namely (1 3 9 11 17 19 25 27)

[1] This was mentioned to Norman Carey by Mark Wooldridge [3], chap. 3.

[2] This case was suggested by David Clampitt in a private communication; it also appears in [8].

– geometric progressions being quite dissimilar in that respect from arithmetic progressions[3]. Owing to this galore of different cases, it is high time the question of the number of generators of a scale was clarified once and for all.

The present paper sets itself to state and prove the known cases: first if a generator is coprime with c, which implies two generators only, except with the 'almost full scale' and its $\Phi(c)$ generators; next classify the remaining cases (including regular polygons), when some generator is not coprime with c; it will be seen that almost any number of generators can occur. A related result involving complementation will be stated. Then I will endeavour to bring the question into a broader focus, first with partial periodicity and application to periodical ME sets, then with Generalized Interval Systems especially in the context of Q-cycles. Lastly, for the sake of completeness, two results on non integer generators will be given.

Notations and conventions. Unless otherwise mentioned, computations take place in $\mathbb{Z}_c$, the cyclic group with c elements. $a \mid b$ means that a is a divisor of b in the ring of integers. The word 'scale' is used, incorrectly but according to custom, for 'pc-set', i.e. a subset of $\mathbb{Z}_c$: no ordering of notes is called for. A 'generated scale' is a subset of $\mathbb{Z}_c$ made of the values of some finite arithmetic sequence (modulo c), e.g. $A = \{a, a+f, a+2f \dots\}$). 'ME set' stands for 'Maximally Even Set', 'WF' means 'Well Formed', 'DFT' is 'Discrete Fourier Transform', and a 'GIS' is short for 'Generalized Interval System'. Φ is Euler's totient function, i.e. $\Phi(n)$ is the number of integers lesser than n and coprime with n.

1. Number of generators of a generated scale

1.1. **The simpler case.** For any (arithmetically) generated scale *where the generator is coprime with c*, there are only two generators, except for the extreme cases mentioned in the foreword:

Theorem 1.
Let $1 < d < c-1$; the scales $A = \{0, a, 2a, \dots (d-1)a\}$ and $B = \{0, b, 2b, \dots (d-1)b\}$ with d notes, generated in $\mathbb{Z}_c$ by intervals a, b, coprime with c, cannot coincide (up to translation) unless $a = b$ or $a + b = c$ (i.e. $b = -a$).

If $d = c - 1$ then there are exactly $\Phi(c)$ generators, the integers coprime with c.

As the result is stated up to translation, the starting points of both scales are irrelevant. The hypothesis can be weakened to *one* generator coprime with c, as it will transpire later.

This has been stated as a fact in several papers, with phrasings like 'the only other generator is $c - a$', though without a formal and general proof, except in particular cases (for instance Well Formed Scales); but what with so many exceptions (see the foreword), a convincing proof ought to be given.

The following proof hangs on the one crucial concept of (oriented) interval vector, that is to say the multiplicities of all intervals inside a given scale. This is best seen by transforming A, B into segments of the chromatic scale, by way of affine transformations.

Proof. All computations are to be understood modulo c. The extreme case $d = c - 1$ was discussed before: any f coprime with c generates the whole $\mathbb{Z}_c$, i.e. $\{f, 2f, \dots (d-1)f\}$ is always equal to $\mathbb{Z}_c$ deprived of 0. Furthermore, a generator **not** coprime with c would only generate (starting with 0, without loss of generality) a strict subgroup of $\mathbb{Z}_c$, hence a subset with stricly less than $c - 1$ elements. So we are left with the general case, $1 < d < c - 1$.

[3]Such geometric sequences occur in Auto-Similar Melodies [2], like the famous initial motive in Beethoven's Fifth Symphony, autosimilar under ratio 3.

Let $D = \{0, 1, 2, \dots d-1\}$; assume $A = B + \tau$, as $A = aD$ and $B = bD$, then D must be its own image – $\varphi(D) = D$ – under the following affine map:

$$\varphi : x \mapsto b^{-1}(a\,x - \tau) = b^{-1}ax - b^{-1}\tau = \lambda x + \mu.$$

We now elucidate the different possible multiplicities of intervals between two pc's in D.

Lemma 1. *Let $1 < d < c-1$, the oriented interval vector $V_D(k) = \text{Card}\{(x,y) \in D^2, y - x = k\}$, yields $V_D(k) < d-1$ $\forall k = 2 \dots c-2$; more precisely,*

$$V_D = [V_D(0) = d, V_D(1) = d-1, V_D(2) \leq d-2, \dots V_D(c-1) = d-1]$$

i.e. interval 1 and its opposite $c-1$ are the only ones with multiplicity $d-1$.

Here is a picture of such an interval vector of a chromatic cluster D (fig. 1).

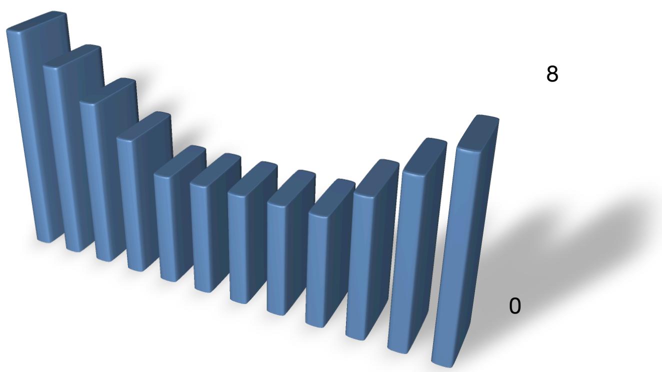

FIGURE 1. Interval vector of (0 1 2 3 4 5 6 7) mod 12.

Proof. Indeed, if we single out an interval $1 < k \leq c/2$ (this is general enough, owing to the symmetry of the interval vector), the pairs of pc's in D which span exactly this interval k are of two kinds (see fig. 2):

$$(1, k+1) \dots (d-k, d) \quad \text{when } k < d, \text{ and, when } d + k > c, \quad (c+1-k, 1) \dots (d, d+k-c),$$

which add up to
$$\begin{cases} d-k & \text{pairs for } k \leq \inf(d, c-d) \\ (d-k) + (d-c+k) = d-(c-d) & \text{pairs for } c-d < k \leq d \\ d-c+k & \text{pairs for } k \geq \max(d, c-d) \end{cases}.$$

In all three cases, the multiplicity is $< d-1$, since k, $c-k$ and $c-d$ are all > 1 by assumption.

For instance, with $c = 12, d = 8, k = 5$ one computes $V_D = [8, 7, 6, 5, 4, 4, 4, 4, 4, 5, 6, 7]$. □

This being stated, the map $\varphi : x \mapsto \lambda x + \mu$ above, which is one to one since λ is invertible in $\mathbb{Z}_c$, multiplies all intervals by $\lambda \mod c$, which turns the interval vector V_D into $V_{\varphi(D)}$ wherein $V_D(\lambda i) = V_{\varphi(D)}(i)$: the same multiplicities occur, but for different intervals. This is a well known feature of affine transformations.

Most notably, the only[4] two intervals with multiplicity $d-1$ are now λ and $-\lambda$. Hence, if $V_{\varphi(D)} = V_D$, the maximal multiplicity $d-1$ must appear in positions 1 and $c-1$, which compels $\lambda = \pm 1$. Finally, as $\lambda = a\,b^{-1} \mod c$, we have indeed proved that $a = \pm b$, qed. □

[4]Because φ is one to one.

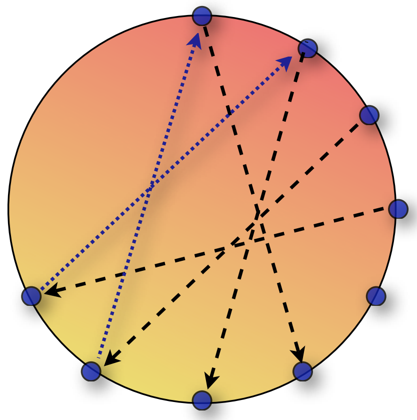

FIGURE 2. Double origin of one interval.

1.2. **Other cases.** So far we have only one case with more that two generators, and that is when the number of notes is too extreme (namely $d = c - 1$ or d), giving $\Phi(c)$ generators. It is time to abandon the condition $\gcd(c, d) = 1$. The difficulty is now that the affine maps $t \mapsto a\,t$, for a not coprime with c, are no longer one to one, so it is not clear how to get back to chromatic clusters like in the proof of Thm. 1.

Let us begin with an interesting generalization of the reasoning in the last proof. It applies to many musically pertinent objects, like the octatonic scale:

Lemma 2. *A scale A with d notes is a reunion of regular polygons, each generated by f, if and only if $V_A(f) = d$.*

Proof. Let us enumerate the starting points in A of interval f, i.e. all $a \in A$ such that $\exists b \in A, a + f = b$. These are all elements of A. This means that $A + f$ is equal to A, hence $a \mapsto a + f$ is a permutation of the set A.

The order of this map is exactly the order of f in group $\mathbb{Z}_c$, i.e. $m = c/\gcd(c, f)$. The orbits of this map are $m-$polygons, which proves the reverse implication. The direct sense is obvious, as each regular polygon provides exactly m times the interval f, as $(a + kf) - a = f \iff (k - 1)\,f = 0$ in $\mathbb{Z}_c$, which implies that $k - 1$ is a multiple of m and hence $a + k\,f = a + f$, the only successor of a in the orbit. □

There are further extensions of this. First notice that a scale featuring an interval f with multiplicity $d-1$, when d divides c, is not necessarily generated by this interval, as this scale can be made for instance of several full orbits of $x \mapsto x + f$, plus one chunk of another orbit. This is quite different of the $\gcd(c, d) = 1$ case of Thm. 1. There are[5] numerous musical occurences of such scales, for instance (2 5 8 9 11) and (1 4 7 10 11) in fig. 3. This will make Thm. 9 that much more significant.

Also,

Lemma 3. *$A \subset \mathbb{Z}_c$, a scale with d notes, is a regular polygon iff $V_A(f) = d$* ***for some divisor f of c****.*

Proof. Same as above, but there is only one orbit, i.e. a regular polygon, the orbit of some a, e.g. $a + f\mathbb{Z}_c$. □

[5]In [1], Online Supplementary 3, I have studied which scales in general achieve the greatest value of their maximum Fourier Coefficient. It turns out that they exhibit this kind of geometrical shape.

FIGURE 3. Minor third with multiplicity 4 in 5 notes, in Liszt's sonata.

In that case, a generator of the scale is the same thing as a generator of the difference group $< A - A >= f\,\mathbb{Z}_c = \dfrac{c}{\gcd(c,d)}\mathbb{Z}_c$ [6]. This is the case pointed out by D. Clampitt[7]. As any subgroup of $\mathbb{Z}_c$ [with d elements] must be cyclic, and has $\Phi(d)$ generators (this being another definition of the totient function Φ) we can state

Theorem 2. *A regular polygon in $\mathbb{Z}_c$, i.e. a subset of the form $a + f\mathbb{Z}_c$, $f = c/d$ being a divisor of c, has exactly $\Phi(d)$ generators.*

Of course, for any such generator, any point of the polygon can be used as starting point for the generation of the scale.

There remains to be considered the case of a scale generated by some f **not** coprime with c, and that is not a regular polygon. For the end of this discussion, let $\gcd(f,c) = m > 1$, and assume $d > 1$ and $0 \in A$ (up to translation).

We introduce $\mathcal{F}_A(t) = \sum_{k\in A} e^{-2i\pi kt/c}$, the Discrete Fourier Transform of A. This is a secret weapon popularized by [8], see [1] for details about the maths.

When $A = \{f, 2f, \dots d\,f\}$ one gets from a simple trigonometric computation

Lemma 4.

$$|\mathcal{F}_A(t)| = \begin{cases} \dfrac{|\sin(\pi dtf/c)|}{|\sin(\pi tf/c)|} & \text{or} \\ d & \text{when } \sin(\pi tf/c) = 0 \end{cases}$$

Moreover, $|\mathcal{F}_A(t)| \le d$ and $|\mathcal{F}_A(t)| = d \iff \sin(\pi tf/c) = 0$.

Proof. The formula is derived from Euler's $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ and the computation of

$$(e^{i\theta} - e^{-i\theta})\sum_{k=1}^{d} e^{-2ki\theta} = e^{-i\theta} - e^{-(2d+1)i\theta} = e^{-i\theta}e^{-d\,i\theta}(e^{d\,i\theta} - e^{-d\,i\theta})$$

The inequality comes from $|\sin(2\theta)| = 2|\cos\theta\sin\theta| < 2|\sin\theta|$ and by easy induction, $|\sin(d\theta)| < d|\sin\theta|$, for integer $d \ge 2$ and $0 < \theta < \pi$ with here $\theta = \pi tf/c \mod \pi$. □

If one can also write $A = a + \{g, 2g, \dots d\,g\}$, then one similarly gets $|\mathcal{F}_A(t)| = \dfrac{|\sin(\pi dtg/c)|}{|\sin(\pi tg/c)|}$ (because $|\mathcal{F}_A|$ does not change when A is translated). Hence, as this quantity cannot reach maximum value d unless both sines are nil, $t\,g/c$ and $t\,f/c$ must get simultaneously integer values, i.e.

[6]This is the group generated by $A - A$, in all generality, cf. [9], 7.26. It is illuminating to visualize this group as a kind of tangent space of A, like the vector space associated with an affine structure.

[7]Private communication.

Lemma 5. *If f, g are two generators of a same scale A, then $m = \gcd(c, f) = \gcd(c, g)$.*

NB: this lemma can be reached algebraically, but it is not altogether trivial.

From there, one can divide A by m and assume without loss of generality $f' = f/m$ and $c' = c/m$ coprime.

But now we have a doubly generated scale $A' = A/m$ in $\mathbb{Z}_{c'}$ with generator(s) f/m and g/m coprime with c': then Thm. 1 gives two cases, either $\operatorname{Card} A' = d < c' - 1$ or not. In the latter case, we have $\Phi(c') = \Phi(d+1)$ generators for an almost full, or full, aggregate; in the former, only two, like for the generic 'major-like' scale.

As for instance, $f' = \pm g' \pmod{c)'} \iff f = \pm g \pmod{c}$, we have the answer at last:

Theorem 3. *A scale generated by f* ***not*** *coprime with c, with a cardinality $1 < d < c$, has*

- *two generators (not coprime with c) when d is strictly between 1 and $c' - 1 = c/m - 1$;*
- *$\Phi(d)$ generators when $d = c'$ i.e. when A is a regular polygon;*
- *$\Phi(d)$ generators when $d = c' - 1$, which share the same order in the group $(\mathbb{Z}_c, +)$;*

where $m = \gcd(c, d)$ and $c' = c/m$.

As we can see on fig. 4, the scales in Thm. 3 are *geometrically* not new, since they are enlarged versions of the previous cases (those of Thm. 1) immersed in larger chromatic universes.

The last, new case, features incomplete regular polygons, i.e. regular polygons with one point removed. For any divisor c' of c, let $f = c/c'$ and $d = c' - 1$:

$$\{f, 2f, \dots d\,f = c - f = -f \pmod{c}\}$$

is the simplest representation of such a scale. All others are translates of that one. (1 3 5 7) in ten-note universe, mentioned in the foreword, was one of them.

Consideration of the 'gap' in such scales would yield an amusing fact, apparent in this example and whose proof will be left to the reader:

Proposition 1. *For each generator of such an 'incomplete polygon', there is a different starting point.*

For instance, scale (0 2 4 6 8 10 12) in $\mathbb{Z}_{16}$ has the $\Phi(8)$ generators 2, 6, 10, 14 with starting points 0, 4, 8, 12. See fig. 4, noticing that both paths can be reversed.

Summing up, the number of generators of a scale can be anything from 2 to $\Phi(c)$ with all the $\Phi(c'), c' \mid c$, in between. The geometry of generated scales comes in three types:

- Regular polygons,
- Regular polygons minus one note,
- 'major scale-like scales', i.e. scales with only two (opposite) generators.

So this seminal last case is by no means the only one.

1.3. **Generatedness and complementation.**

There is a rather weird sequel of these theorems, already published in the case of Maximally Even Sets ([1]):

Theorem 4 (Chopin's theorem for generated scales)**.**

If two generated scales A, B are complementary, then some translate of one of them is a subscale of the other.

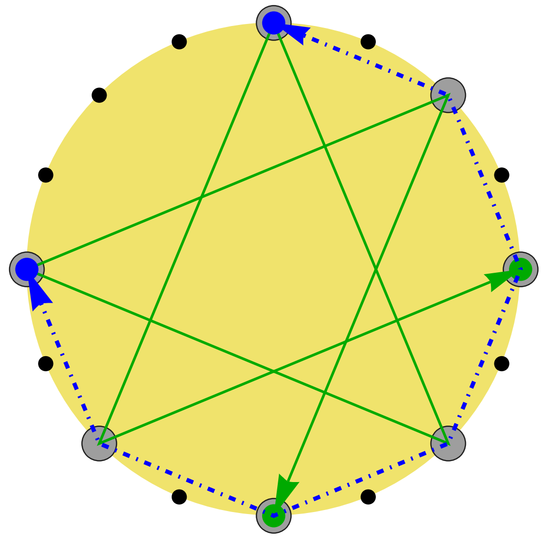

FIGURE 4. Four generators for this scale

This stands for non trivial scales, that is to say both have at least two notes. An example is (C D F G) and its eight notes complement, in twelve-tone universe.

Proof. First we study the case when one generating interval is coprime with c; let for instance $A = \{f, 2f, \dots d\,f\}$. Then $A' = \{(d+1)f, \dots cf\}$ is the complement of A in $\mathbb{Z}_c = \{f, 2f, \dots cf\}$. Hence $A' = B$, and A, B share a generator: either $B - d\,f \subset A$, or the reverse.

Lastly, let us assume that both A and B have no generator coprime with c. Then A (and B likewise) is, up to translation, a subset of some subgroup of $\mathbb{Z}_c$, hence has less than $c/2$ elements. More precisely, if (say) $A = \{0, f, \dots (d-1)f\}$ then A is a subset of $\gcd(c, f)\mathbb{Z}_c$, subgroup with $c' = c/\gcd(c, f)$ elements. Unless both cardinals are $c/2$, $A \cup B$ cannot be equal to the whole $\mathbb{Z}_c$. Thus the only case whence A can still be the complement of B is when c is even, and A, B are complement halves (like for instance the two whole-tone scales), which means $\gcd(c, f) = 2$, and 2 is a generator. In that case $A = B + 1$, which completes the proof. □

Remarks. *I called this Chopin's theorem, as it generalizes the construction used by Chopin in opus 10, N. 5 etude in G flat major, where the right hand plays only black keys (e.g. a pentatonic scale, which is a Maximally Even set generated by fifths) and the left hand plays in several (mostly) major scales*[8] *(another ME set generated by fifths), each of which includes the black keys. In that situation*[9] *it has certainly been observed before (*[1]*), but as a statement on general generated scales it is new, as far as I know, though it could easily be derived from the Maximally Even case.*

2. Broader Perspectives

The point that this paper purported to clear has actually been evoked in the literature, but always partially (in some particular context, e.g. Well Formed Scales), or without proof. Now that we have given a complete liste and a detailed proof of its completeness, it is worthwhile to look again at the question of generators from some broader perspectives.

[8]Three major scales include the five black keys.

[9]And even in a more general setting, with the subsets that Ian Quinn calls 'prototypes', which form a class invariant by complementation [8].

2.1. **Generatedness and periodicity.** Informally, a generated scale is a chunk of a periodical sequence. So it should be recognizable under harmonic analysis[10]. Indeed, there exists an alternate proof of Thm. 1, using the Discrete Fourier Transform, that pinpoints the degree of periodicity of a subset of $\mathbb{Z}_c$; the same DFT[11] was instrumental in the case of a generator not coprime with c, cf. Lemma 5. One could single out the generator of a generated scale by the maximum size of a Fourier coefficient:

Theorem 5. *For c, d coprime, a scale with d notes is generated by an interval f, coprime with c, if and only if the semi-norm*

$$\|\mathcal{F}_A\|^* = \max_{t \text{ coprime with } c} |\mathcal{F}_A(t)| = \max_{t \text{ coprime with } c} \Big|\sum_{k \in \mathbb{Z}_c} e^{-2i\pi\, k\, t/c}\Big|$$

is maximum among all d-element scales. Moreover, if $\|\mathcal{F}_A\|^* = |\mathcal{F}_A(t_0)|$, then t_0^{-1} is one generator of scale A, the only other being $-t_0^{-1}$.

This extends the discovery made by Ian Quinn [8] that A is Maximally Even with d elements if, and only if, $|\mathcal{F}_A(d)|$ has maximum value among d elements subsets; it is also another illustration of his philosophy of looking at scales through their DFT. We have elaborated on this in [1], wherein this last theorem is proved (Online Supplementary III).

In a nutshell, this hinges on the two following facts:

- Any affine transformation permutes the Fourier coefficients ($\mathcal{F}_{k\,A}(t) = \mathcal{F}_A(k\,t)$ for any k coprime with c)
- $|\mathcal{F}_A(1)|$ is maximum when all elements of A are consecutive (A is a chromatic cluster).

The relationship between this and the interval-vector proof of Thm. 1 might be that the DFT of the interval vector is the squared absolute value of the DFT of the characteristic function of the scale ([1],[5]).

Let us now consider Maximally Even Sets with d elements, where c, d are not coprime; these are not generated scales, but they are build from a generated cell that is repeated periodically; the maximums of the DFT will be perforce related to those different generators of the subscales that generate the ME set by periodicity, by the principle of heredity in ME sets. As the structure of these ME sets is well known, it will suffice to give an example, state a general result (see [1] for a proof), and draw a nice figure.

In the diatonic universe ($c' = 7$), there are 7 maximally even triads, which are the translates of $A' = (0\ 2\ 4)$[12]. Their two generators are 2 and 5. If we consider now a $c = 28 = 4 \times c'$ note chromatic universe, the $4 \times 3 = 12$-note ME sets are obtained by periodicity from the triads; here is one out of these seven ME-sets:

$$A = ME(28, 12) = ME(7, 3) \oplus 7\,\mathbb{Z}_{28} = (0, 2, 4) \oplus (0, 7, 14, 21) = (0, 2, 4, 7, 9, 11, 14, 16, 18, 21, 23, 25)$$

The maximums of the DFT of A are obtained for indexes $(3, 4) \times 4 = (12, 16)$, while the maximums of the DFT of A' are obtained for 3, 4, which are the inverses mod 7 of generators 2, 5. The relationship between the Fourier Transforms of both ME sets, A and A', is obvious on figure 5.

The generalization of this example stands as follows:

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[10] In the mathematical acception.

[11] The DFT of a subset of $\mathbb{Z}_c$ is simply the DFT of its characteristic function, as defined just before Lemma 4.

[12] Caution: there is no need to embed the diatonic into the chromatic; if it were done, then these triads would come in 3 shapes – "cardinality equals variety" – though in $\mathbb{Z}_7$ they form but one class under translation.

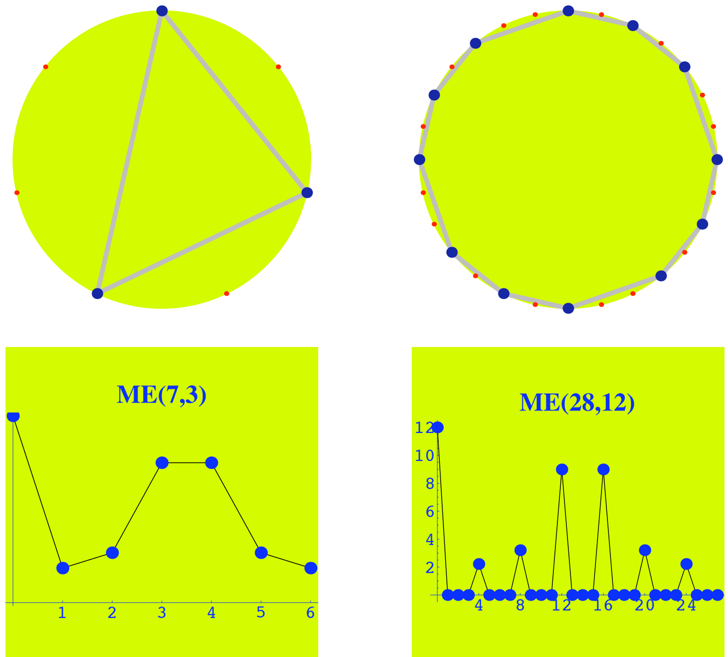

FIGURE 5. The maximum DFT coefficients of ME(28,12) and ME(7,3).

Theorem 6. *Let A be a ME set in $\mathbb{Z}_c$ with d elements, $1 < m = \gcd(c, d) < d$. The two generators of the ME set A' with $d' = d/m$ elements in $\mathbb{Z}_{c'}, c' = c/m$, that in turn generates $A = A' \oplus c'\mathbb{Z}_c$ by c' periodicity, are the multiplicative inverses modulo c' of the two indexes of the maximums of $|\mathcal{F}_A|$, after they are turned into elements of $\mathbb{Z}_{c'}$ by a division by m. In consequence, there are precisely $2m$ ways to define*

$$A = \{f, 2f, \dots d'f\} \oplus c'\mathbb{Z}_c :$$

f is equal to $\pm d'^{-1} \mod c'$, plus any multiple of c'.

The last sentence might seem weird but makes sense. Consider $A = \{1, 2, 4, 5, 7, 8, 10, 11\}$ for $c = 12 : d = 8, m = 4, d' = 2, c' = 3$: one generates the cell $A' = \{1, 2\}$ in $\mathbb{Z}_3$ from either generator $f' = 1$ or 2, and any value $f \in \mathbb{Z}_{12}$ congruent to f' modulo 3, eg any element of $\mathbb{Z}_{12} \setminus 3\,\mathbb{Z}_{12}$, 'generates' A in the sense of the formula above.

Proof. See [1], supplementary III, for a discussion of the extrema of Fourier coefficients when $m > 1$; the theorem ensues. Alternatively, this is a consequence of the well-known structure of ME sets when $1 < m = \gcd(c, d) < d$ and of Thm. 1. □

When d is a divisor of c, i.e. $m = \gcd(c, d) = d$ and the ME set degenerates, i.e. it is a regular polygon, then the theorem above fails, as there are $\Phi(d)$ generators but c/d maximums (they are the multiples of d, 0 included), cf. Thm. 7 below.

2.2. **Generatedness and cyclic GIS.** Another point of view is the group-theoretic one. Remembering Lewin's approach of intervals as operators, a generated scale can be viewed as the effect (more precisely: the orbit, or some truncation of it) of some translation $x \mapsto x + a$. Now invariance of this orbit under some multiplication (recall the proof of Thm. 1) would mean that it can be alternatively generated by a conjugate map under some affine transformation $x \mapsto k\,x$. Such a transformation turns translation $\tau_a : x \mapsto x + a$ into $\tau_b : x \mapsto x + b$, where $b = k\,a$ has exactly the same order as a in the additive group $\mathbb{Z}_c$[13]. This enables us to retrieve the case of

[13] It being the order of the maps τ_a and τ_b in the affine group.

regular polygons and their different generators, which are characterized by their order in the group $\mathbb{Z}_c$. Though this case has already been mentioned in Thm. 3, let us reformulate it here in another light[14]:

Theorem 7. *The number of generators of a regular polygon[15] in $\mathbb{Z}_c$, i.e. one full orbit of a translation τ_f, is the number of generators of the subgroup $f\,\mathbb{Z}_c$, i.e. $\Phi\big(\frac{c}{\gcd(c,f)}\big)$.*

Remarkably, there is no such polygon in $\mathbb{Z}_{12}$ with more than 2 generators (except the whole chromatic aggregate), because for all divisors d of 12, $\Phi(d) \le 2$. Perhaps this explains why the falsity that "a generated scale has no more than two generators" has long been taken for granted'.

This may appear as an excessively complicated way to compute that the generators of a regular polygon scale must have the same order; but affine classes, orbits, conjugacy classes are useful tools for the study of pc-sets, and it is worthwhile to practice with them. This also puts generated scales in the powerful general framework of Generalized Interval Systems: as David Clampitt noticed already in [4], generated scales are members of Q-cycles[16] with a cyclic group of translations, in the sense of Lewin's GIS (see [7])[17]. Without unrequited technicalities, this means here that translating by f a scale generated by f will give another scale with exactly one different note – in the case of the major scale this explains the generation of key signatures.

Now all generated scales sharing the same generator f and number of notes (e.g. all Major Scales) are the members of one Q-cycle of scales, at least when c, d are coprime, e.g. orbits in the set of subsets of $\mathbb{Z}_c$ of the cyclic group of translations $f\mathbb{Z}_c$, which acts simply transitively on scales. Such a Q-cycle forms a GIS with the interval group generated by f.

So the question of different generators can be extended to a rather more abstract one: if the scale $S \subset \mathbb{Z}_c$ is part of some Q-cycle, meaning $S + f$ is S with but one note changed, is it part of another Q-cycle with a different f ? Clampitt addresses a much more general (and difficult) question in the last but one chapter of [4], namely the different GIS that give rise to some definite (e.g. pairwise WF) scale. Here we consider only cyclic groups of translations, acting on pitch classes. Thm. 1 answers that there are only two ways to cycle around the Q-cycle of a non degenerate well-formed scale, Thm. 3 gives two or more, and Thm. 7 more than two. In Clampitt's framework however, our search for different generators appear as part of the honourable quest for all meaningful (e.g., group-theoretic) interpretations of a scale.

2.3. **Non integer generation.** The present paper covered exhaustively the case of scales of the form $\{k\,f \pmod c, k = 0 \dots d-1\}$; but in their own way, non integer generators are also common, for instance in the case of the $J-$functions that generate Maximally Even Sets:

$$J_\alpha(k) = \lfloor k\alpha \rfloor \pmod c, k = 0 \dots d-1, \quad \text{wherein usually } \alpha = \frac{c}{d}$$

Of equal importance (it can be construed as a generalization of the above) are sets of values of the maps $k \mapsto \mathcal{P}_x(k) = k\,x \pmod 1$ that are useful for pythagorean-like scales (e.g. $x = \log_2(3/2)$) and other Well Formed scales. The symbol $\lfloor t \rfloor$ denotes the floor function, i.e. the greatest integer lower than, or equal to t. The question of different generators can be formulated thus:

[14] I believe D. Clampitt was the first to single out this case.

[15] Called 'Degenerate Well Formed' by Carey & Clampitt.

[16] After Clough and Cohn, we call P-cycle a subset of $\mathbb{Z}_c$ that admits some translate equal to itself but for one note, which moves by one step unit. The example of the Major Scale springs to mind. Q-cycles are P-cycles with the last condition dropped, e.g. (C D E)→ (D E F♯) →. . . .

[17] A GIS – Generalized Interval System – is essentially a collection of objects together with a group that enables to go to and fro between the objects, without redundancy. See [6] for a very readable introduction.

If two sets of values of J_α, J_β (resp. $\mathcal{P}_x$, $\mathcal{P}_y$) are transpositionally equivalent, do we have $\alpha = \pm\beta$ (resp. $x = \pm y$)?

We have already seen that the general answer is *no*, for instance when $\alpha \in \mathbb{N}, d\alpha = c$ (or similarly $d\,x \in \mathbb{N}$). Other cases are worth investigating.

Let us first consider the values of a J function with a random multiplier, e.g. $J_{a/b}(k) = \lfloor k\,a/b \rfloor \pmod c$. These values have been mostly scrutinized when $a = c, b = d$, for the generation of (c, d) ME-sets, but there is no law against producing a scale with consecutive values of such a function for any given (real) α.

Theorem 8. *d consecutive values of such a J_α function (up to translation) do not characterize the ratio α.*

Proof. A counter-example will suffice. For instance, $\alpha = 37/22, d = 7$ yield the same scale as $\alpha = 12/7$ (A flat major). It is understandable that such an approximate formula allows some leeway in the choice of the ratio α: one could notice that the scale produced by J_α is locally constant in the variable α. Of course, the *infinite* sequence $J_\alpha(k), k \in \mathbb{Z}$ would provide much more information on α. Another possible restriction would be an upper bound for the denominator of α: this could coerce $\alpha = \pm c/d$.

□

As a final effort, I will state one result when the generator, in the $\mathcal{P}_x$ construction, is irrational:

Theorem 9. *If the sets $S_x = (0, x, 2x, \dots (d-1)x) \pmod 1$ and $S_y = (0, y, 2y, \dots (d-1)y) \pmod 1$ are transpositionally equivalent, with x irrational, then $x = \pm y \pmod 1$.*

This is implicitely known for WF scales in non tempered universes, but this theorem is more general.

Proof. It follows the idea of the proof of Thm. 1. This works because the affine map $\mathcal{P}_x$ is one to one again:

Lemma 6. $\forall a, b \in \mathbb{Z},\ a\,x \equiv b\,x \pmod 1 \iff a = b.$

Proof. Else x would be rational. □

Consider all possible intervals in S_x, i.e. the $(i-j)x \pmod 1$. By our hypothesis, these intervals occur with the same multiplicity in S_x and S_y. Let us have a closer look at these intervals (computed modulo 1), noticing first that

- There are d different intervals from 0 to $k\,x$, with $k = 0$ to $d-1$. They are distinct because x is irrational, as just mentioned. Their set is $\mathcal{I}_0 = (0, x, 2x \dots (d-1)x)$.
- From x to $x, 2x, 3x, \dots (d-1)x$ *and 0*, there are $d-1$ intervals common with I_0, and a new one, $0 - x = -x$. It is new because x is still irrational. For the record, their set is $\mathcal{I}_x = (0, x, 2x \dots (d-2)x, \boldsymbol{-x})$.
- From $2x$ to the others, $d-1$ intervals are common with $\mathcal{I}_x$ and only $d-2$ are common with the $\mathcal{I}_0$.
- Similarly for $3x, 4x \dots$ until
- Finally, we compute the intervals from $(d-1)x$ to $0, x, \dots, (d-2)x, (d-1)x$. One gets $\mathcal{I}_{(d-1)x} = (0, -x, -2x \cdots - (d-1)x)$.

The following table, not unrelated to fig. 1, will make clear the values and coincidences of the different possible intervals:

starting										
0					0	x	2x	...	(d-2)x	(d-1)x
x				-x	0	x	2x	...	(d-2)x	
2x			-2x	-x	0	x	2x	...		
⋮										
(d-1)x	-(d-1)x		...	-x	0					

(note for reviewers: if preferred, I have a coloured version of the table here:)

Intervals starting in										
0					0	x	2x	...	*(d-2)x*	*(d-1)x*
x				-x	0	x	2x	...	*(d-2)x*	
2x			-2x	-x	0	x	2x	...		
...										
(d-2)x		*-(d-2)x*	...	-x	0	x				
(d-1)x	*-(d-1)x*	...	...	-x	0					

FIGURE 6. The different intervals from each starting point

So only two intervals (barring 0) occur $d-1$ times in S_x (resp. S_y), namely x and $-x$. Hence $x = \pm y$, qed. □

CONCLUSION

Apart from the seminal case of Major Scale-like generated scales, it appears that many scales can be generated in more than two ways. This is also true for more complicated modes of 'generation'. I hope the above discussion will shed some light on the mechanics of scale construction. I thank David Clampitt for fruitful discussions on the subject, and Ian Quinn who edged me on explore it in depth.

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