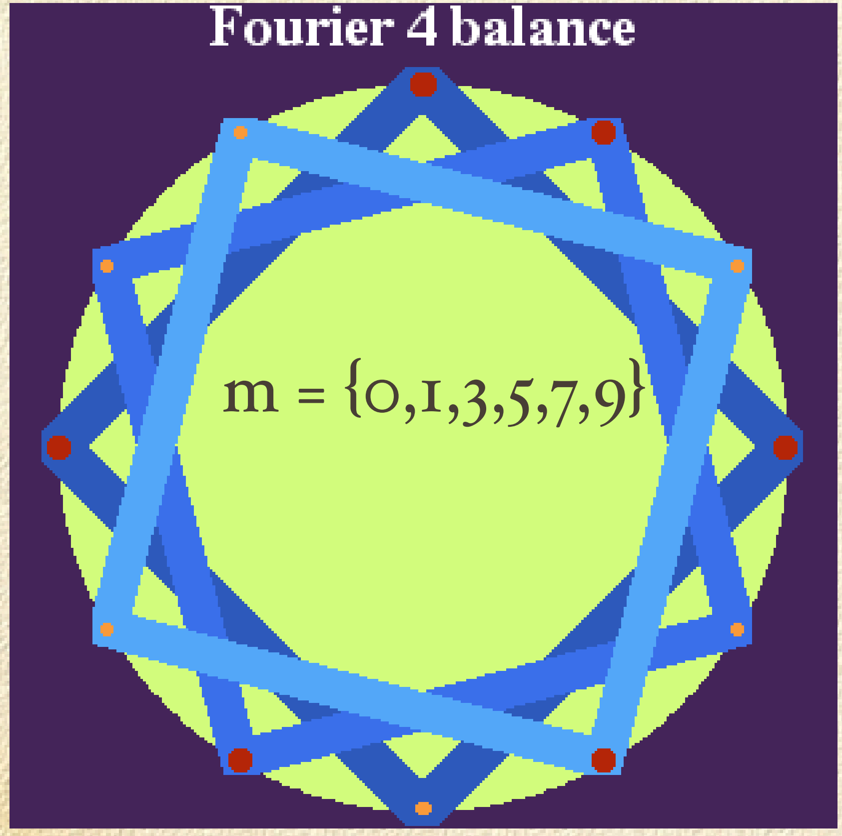
‘Round Fourier

A journey in harmonic analysis of pc-sets

Scales & balances

- ‘Fourier balances’ : a view on the inner symmetry of chords



Lewin's call for Fourier

- Fourier = the hidden periods inside something
- Enhancing the structure : algebra
  - the interval function is messy (convolution of characteristic functions), but turns into a simple product when 'Fourierized.

$$1_A * \tilde{1}_B(k) = \sum_i 1_A(i) \times 1_B(i-k) = \sum_{\substack{i\in A \\ i-k\in B}} 1$$

$$\mathcal{F}(1_A * \tilde{1}_B) = \mathcal{F}(1_A) \times \mathcal{F}(\tilde{1}_B)$$

Lewin's call for Fourier: an example

{0,1,4,5,8}

For each pc, set a wheel in motion

[redacted]

[redacted]

Lewin's call for Fourier: an example

{0,1,4,5,8}

Values for *t*=1,2,3,4,5,6 give Lewin's properties

[redacted]

This vanishes when there are as many odd as even pc's:
this the whole-tone property, or Fourier 6 in the 2001 paper

Lewin's call for Fourier: an example

{0,1,4,5,8}

A measure of Imbalance
on Fourier balance 6

[redacted]

The imbalance of set *m* for Fourier balance *d* is precisely
$|\text{fourier}(m, d)|$

Hence the connection between
Lewin's work and Clough and Douthett's :

$|\text{fourier}(d\text{-MEset}, c/d)| \approx d \geq |\text{fourier}(\text{any } d\text{-set, any } t)|$

Fourier as a powerful tool

- Vuza and RCMC : proving Hajòs theorems (1990)
  - Characterization of *some* subsets: tiling, with no regularity
- Lagarias and Wang's theorem (1996)
  - A harder version of a theorem by Vuza's, uses difficult results on zeroes of Fourier series
- Babbitt's hexachord theorem is a one-liner with Fourier
- Similarly, explains why complement of ME set is ME too (more or less: their Fourier transforms are opposite) (perhaps also thm 3.1).
- Wild's Trichords Give Palindroms

Fourier as a criterion

- Fuglede's spectral conjecture (1974)

A spectral set: {0,3,5,10}

$$\text{fourier}\ (sp, t) = 1 + e^{\frac{i\pi t}{2}} + e^{\frac{5i\pi t}{6}} + e^{\frac{5i\pi t}{3}}$$

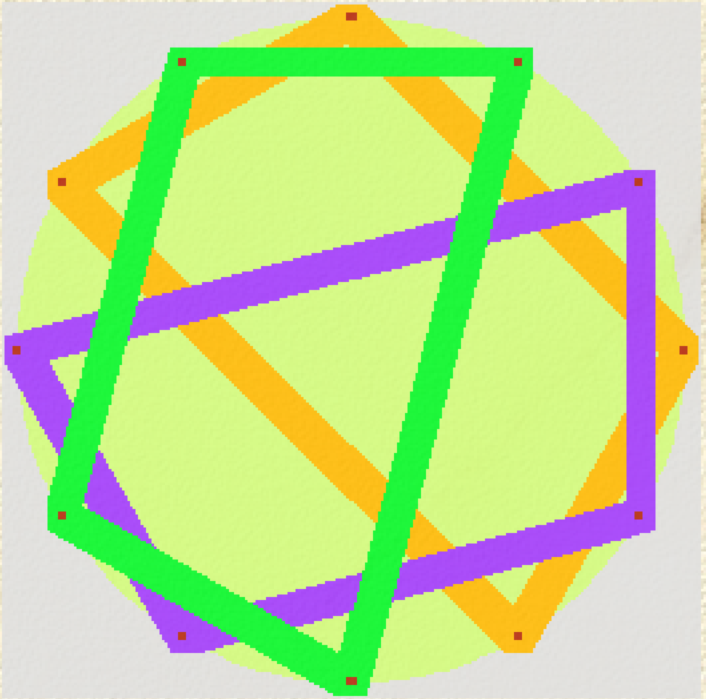
The spectral condition holds :

$$\text{diff} = \{0, 3, 6, 9\} - \{0, 3, 6, 9\} = \{-9, -6, -3, (0,)\, 3, 6, 9\}$$

$$\text{diff}\ //\ \text{fourier}\ (sp, t) = \{0, 0, 0, 0, 0, 0\}$$

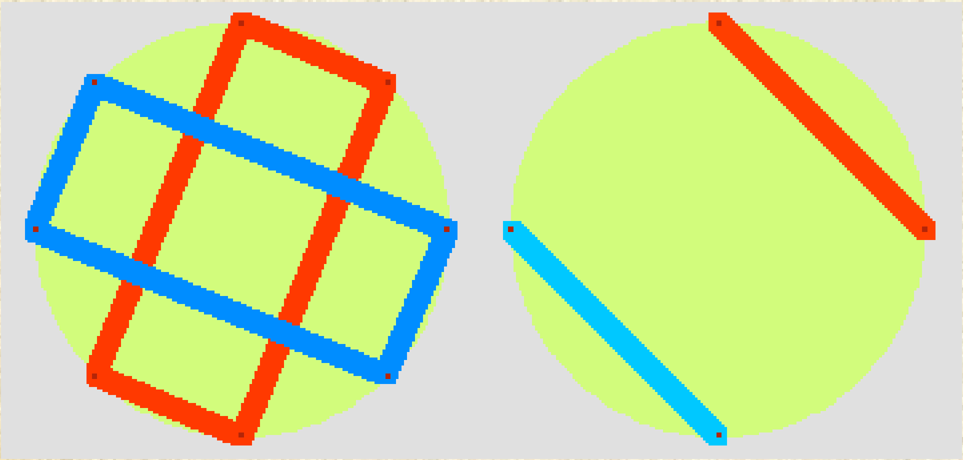
Fourier as a criterion

- Fuglede's spectral conjecture
  - tiling <=> 'spectral'
  - mostly (and probably) true
  - generated by class IIa pc-sets



An aside on the tiling property

- Very often a tiling 'chord' reduces to a smaller one in a smaller universe.
- cf. R. Cohn's cycles
- This means $t \rightarrow \text{fourier}(m, t)=0$ is p periodic for some $p<n$.



Return to Fuglede: the conjecture is false !

- Counter example : Terence Tao, 2003
- The universes are products of cyclic groups (e.g. $(\mathbb{Z}/3\mathbb{Z})^6$)
- Not unfamiliar ground (G.I.S.) for Lewin's fans

Hadamard matrices

- Counter-examples use Hadamard matrices
  - The columns are mutually orthogonal
- Cf. Lewin's balances

$$\textbf{hadamard} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & e^{\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} & e^{\frac{4i\pi}{3}} \\ 1 & e^{\frac{2i\pi}{3}} & 1 & e^{\frac{4i\pi}{3}} & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} \\ 1 & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} & 1 & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} \\ 1 & e^{\frac{4i\pi}{3}} & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} & 1 & e^{\frac{2i\pi}{3}} \\ 1 & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} & 1 \end{pmatrix};$$

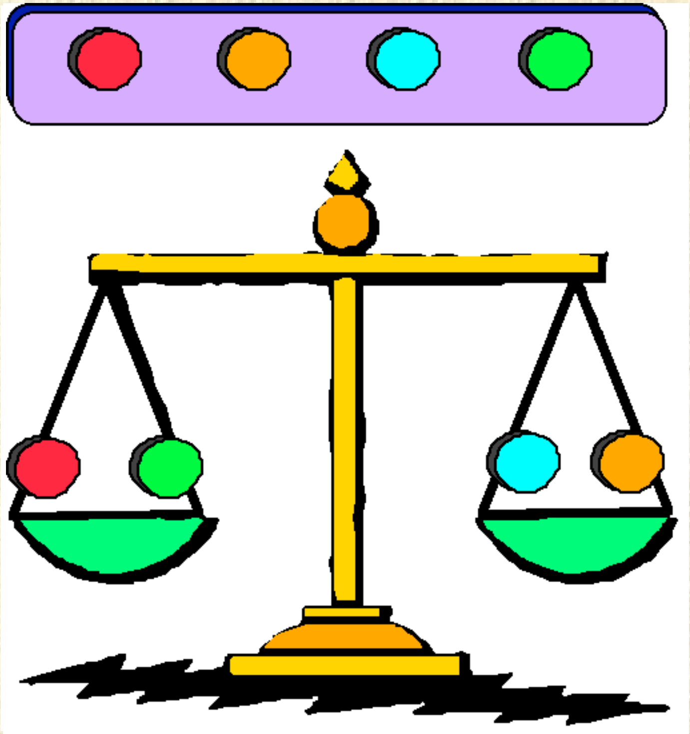
Simplify[hadamard.hadamard†]

$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

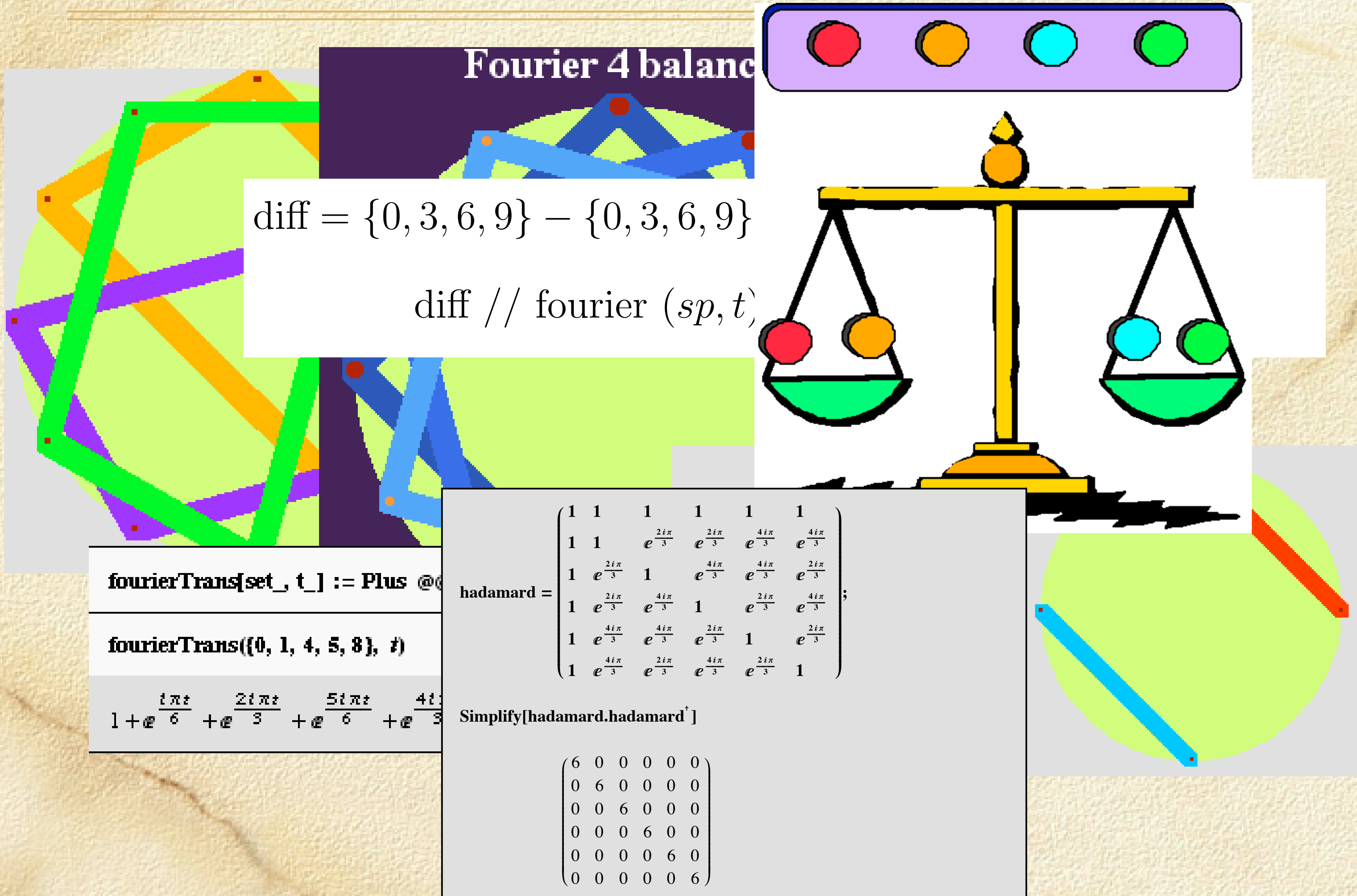
Back to square one

- Hadamard matrices : originally intended for weighing problems !
- Balancing true and forged coins

$$\mathbf{had} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$



The end


Fourier 4 balanc
diff = {0, 3, 6, 9} − {0, 3, 6, 9}
diff // fourier (sp, t)
fourierTrans[set_, t_] := Plus @@
fourierTrans[{0, 1, 4, 5, 8}, t]
hadamard =
Simplify[hadamard.hadamard†]

Never, never misunderestimate Fourier !