Evanston, oct. 24th

Rhythmic Canons, Galois Theory, Spectral Conjecture

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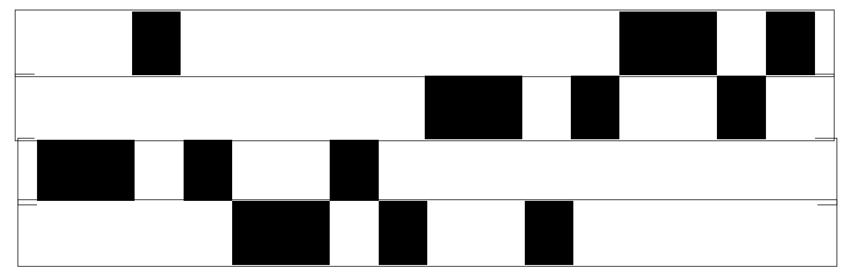
Rhythmic Canons

- What is a rhythmic canon ?
- Mathematical tools
- Canons modulo p
- Transformation, reduction, conservation

Rhythmic Canons

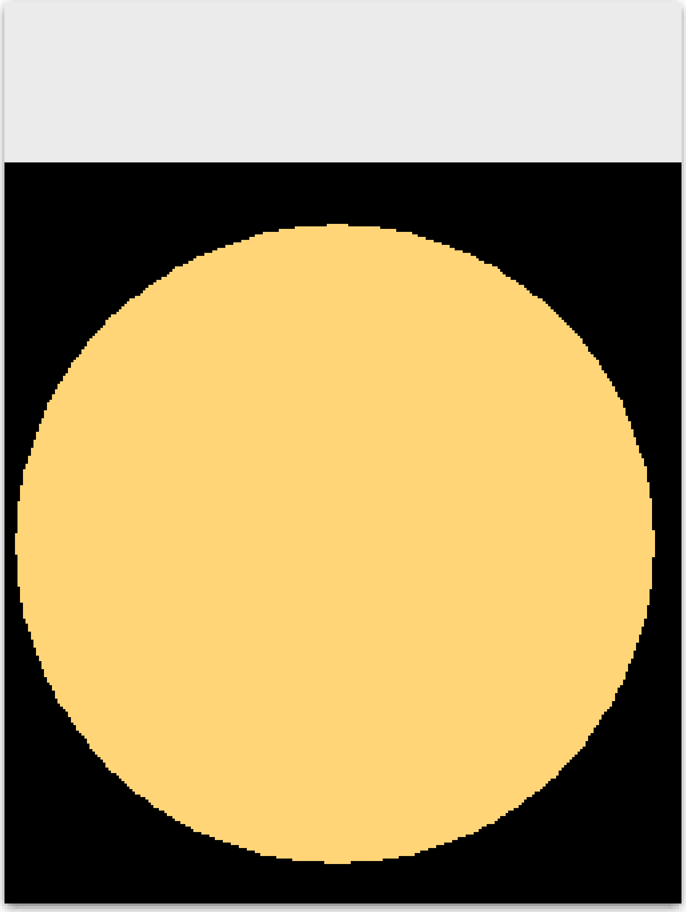
- A canon is a number of voices playing the same tune at different onsets.
- A rhythmic canon is a number of voices playing repeatedly the same rhythmic pattern at different onsets.

A rhythmic canon is periodic (here modulo 16)



Rhythmic Canons

- The rhythmic pattern is called **inner voice**,
- The set of onsets is the **outer voice**
- Together they **tile a cyclic group**



Maths for canons

- A direct sum $$A \oplus B = \mathbb{Z}/n\mathbb{Z}$$
- Exponentiation $$A(x) = \sum_{k \in A} x^k$$
- Condition (T_0)

$$A(x) \times B(x) = (A \oplus B)(x) \equiv 1 + x + \ldots x^{n-1} \pmod{X^n - 1}$$

$$\{0, 1, 3, 6\} \oplus \{0, 8, 12, 4\}$$

$$(X^6 + X^3 + X + 1)\,(X^{12} + X^8 + X^4 + 1)$$

$$X^{18} + X^{15} + X^{14} + X^{13} + X^{12} + X^{11} + X^{10} + X^9 + X^8 +$$
$$X^7 + X^6 + X^5 + X^4 + X^3 + X + 1$$

Some Rings

0-1 polynomials

$\{0,1\}[X]$ is not a ring

$(1+X+X^2)(1+X^2)=???$

$\mathbb{Z}[X]$ is too big

$\mathbb{F}_2[X]$ is weird

$(1+X+X^2)(1+X^2)=$
$1+X+2X^2+2X^3+X^4$

$(1+X+X^2)(1+X^2)=$
$1+X+X^4$

Where does (T_0) make sense ?

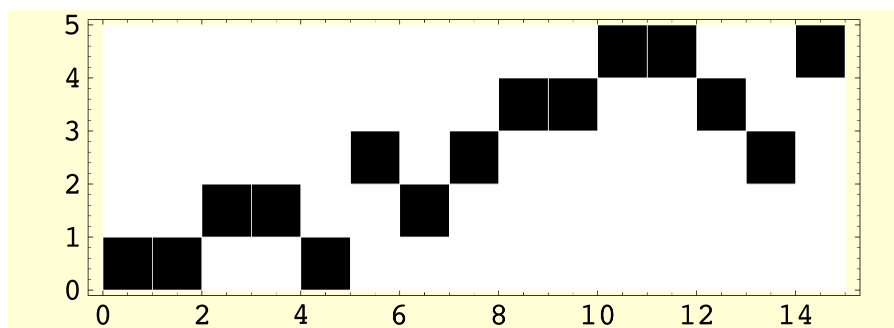
- 0 and 1 are elements of any field
- 'Tiling modulo p' means '(T_0) holds in $\mathbb{F}_p[X]$'

Chinese rhythmic canon theorem (2002):
If $A(x)\ B(x) = 1 + x + \ldots x^{n-1} \bmod x^n - 1$
in all $\mathbb{F}_p[X]$, then it holds in $\mathbb{Z}[X]$.

Galois theory in $\mathbb{F}_q$

- First occurence : Johnson's problem
- {0 1 4} **and its augmentations** tile with period a multiple of 15, because $1+X+X^4$ splits in $\mathbb{F}_{16}$

Theorem 1
(december 2001)



Several other cases suggested following question:

Is there a 'local to global' approach for the general tiling problem ?

Galois theory in $\mathbb{F}_p$

NO !

Theorem 2 (april 2004)

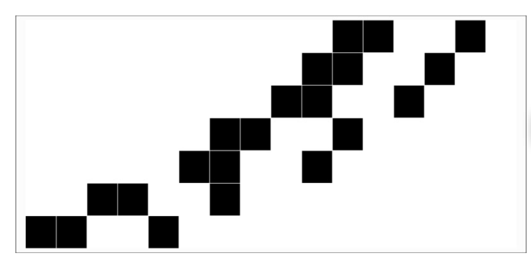
For any finite (non empty) subset $A \subset \mathbb{N}$,

for any prime p, *there exists* $B \subset \mathbb{N}, n \in \mathbb{N}^*$

$$A(X) \times B(X) \equiv 1 + X + X^2 + \dots X^{n-1} \pmod{X^n - 1, p}$$

«Any rhythmic pattern makes a canon
— modulo p»

Example with 0 1 4 :



Conditions (T_1) and (T)

- Remember $A(X).B(X)=1+X+\ldots X^{n-1} \bmod X^n - 1$.
- Cyclotomic factors : irreducible factors of $1+X+\ldots X^{n-1}$ must divide $A(X)$ or $B(X)$. They are the Φ_d, $d \mid n$.
- Let $R_A = \{d \,;\ \Phi_d | A(X)\}$, $S_A = \{p^\alpha \in R_A\}$.

$$(T_1) : A(1) = \prod_{p^\alpha \in S_A} p$$

$$(T_2) : \text{if } p^\alpha, q^\beta, \cdots \in S_A \text{ then } p^\alpha.q^\beta \cdots \in R_A$$

Conditions (T_1) and (T_2)

Theorems (1998, Coven-Meyerowitz)

- If A tiles, then (T_1) is true
- If (T_1) and (T_2) are true, then A tiles
- If A tiles and $|A|=p^\alpha q^\beta$, then (T_1) and (T_2) are true.

Also a very special case with 3 prime factors in 2000 (Lagarias-Wang)

Transformations

Concatenation

- Concatenation
- Other transformations
  - duality : A⊕B = B⊕A !
  - dilatation dilatation
  - affine transform
- Useful for classifying and building up new canons (cf. Vuza canons, in a minute)

Conservation

Theorem 3 (2004):

All usual transformations preserve conditions (T_1) and (T_2)

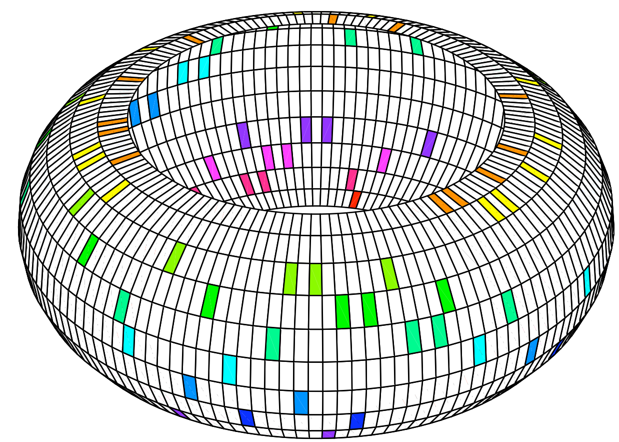
Basic lemma :factorizing the metronome

$1+X^k+X^{2k}+\ldots X^{(p-1)k}$ is the product of the Φ_d whence d is a divisor of n=p k, but not a divisor of k

All this is Galois theory (in cyclotomic fields)

Vuza canons

- Definition: no internal period,
- (unlike (say) {0,1,4,5} + {0,2,8,10})
- Hajòs groups (M.A.) good/bad
- Rather scarse
- Popular with composers



Vuza canons

How do we find them ?

- Difficult to get them all
- Algorithms exist that give a few solutions
- Transformations allow to find much more
- Exhaustive search achieved for n=72 and n=108 (january 2004, H. Fripertinger)

Fuglede's conjecture

Conjecture (Fuglede 1974)

- A set A tiles by translations iff it is spectral (meaning L (A) admits a Hilbert basis)
- True in a number of cases (A convex, set of translations a group…)
- False in high dimension (T. Tao, 2003)

Fuglede's conjecture

- A link with (T_1) and (T_2)
- Theorem : (Isabella Laba 2000)

If A verifies (T_1) and (T_2) then A is spectral.

If A is spectral then (T_1) is true.

Last step

- If A tiles but (T_2) is false,
- If A is not Vuza, then either inner rhythm A or outer rhythm B reduces to a smaller canon ((T_2) still false by theorem 3)
- the process cannot end with the trivial canon ($\{0\} \oplus \{0\}$) - but (T_2) is true here !
- Hence it ends up with a Vuza canon.

Latest news

Theorem 4 (may 2004)

- A canon with (T_2) false can only occur in a non-Hajòs group (and reduces to a Vuza canon)
- **Any tiling of a Hajòs group is spectral**
- This means n  2

From Laba + Coven-Meyerowitz

New

(T_2) is true for a tiling of an interval; checked also by computer in $\mathbb{Z}_{72}$ and $\mathbb{Z}_{108}$

The end ?

- Are all rhythmic canons spectral sets ? this should be found out via cooperation between different fields (musica, algebra, perhaps topology…)
- Both sides ot the Atlantic will be needed.

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