OF ALL INTERVAL TETRACHORDS AND OCTATONIC SCALES

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Abstract
Proving a conjecture of Jason Yust about some products of Fourier coefficients of ME sets.

KEYWORDS: Fourier coefficients, product, Maximally Even Sets.

NOTATIONS

- $\mathbb{Z}_n$ is the cyclic group with n elements.
- for a pc-set A, i.e. a subset of the set of pitches modelled by some cyclic group $\mathbb{Z}_n$, the Fourier transform of A is the sequence of complex Fourier coefficients

$$f_A(t) = f(t) = \sum_{k \in A} e^{-2i\pi kt/n}$$

i.e. the usual Fourier transform of the characteristic function of A.
- $\lfloor x \rfloor$ is the integer part of real x, e.g. $\lfloor \pi \rfloor = 3$.

1. THE CONJECTURE

Working on the correlation of phases (angular directions) of Fourier coefficients of pc-sets (cite a bunch of references), Jason Yust noticed (!) a significant fact that can be best expressed in terms of product of coefficients:

Main conjecture 1. *Consider a Maximally Even set $A \subset \mathbb{Z}_n$ with c elements, and let $a, b \in \mathbb{N}^*$ partition c: $a + b = c$. Then $f(a)f(b)\overline{f(c)}$ is real negative.*

Looking at the phases of coefficients, this means an anti-correlation between the sum of phases of coefficients $f(a), f(b)$ and $f(a+b)$:

$$\Phi_{a+b} = \Phi_a + \Phi_b + \pi$$

if, say, $f(a) = |f(a)|\, e^{i\Phi_a}$.

For example, with $A = \{0, 2, 4, 7, 9\} \subset \mathbb{Z}_{12}$ (a pentatonic collection), taking $a = 3, b = 2, c = 5$ one gets

$$f(3)f(2)\overline{f(5)} = e^{-2i\pi/3}.1.\overline{(2+\sqrt{3})e^{i\pi/3}} = -2 - \sqrt{3}.$$

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2. USEFUL FACTS

The effect of transposition on Fourier coefficients is a rotation in the complex plane. More precisely,

Proposition 1. $f_{A+\tau}(c) = f_A(c) \times e^{-2i\pi\tau c/n}$.

Complementation, on the other hand, turns all coefficients (except the 0^{th}) into their opposites.

The most interesting type of ME sets (including diatonic and pentatonic collections, for instance) occurs when the cardinality is coprime with that of the chromatic universe: $c \wedge n = 1$. Quinn called them *type I ME sets*.

It is proved in [**?**] and other places that such ME sets are *generated*, i.e. up to transposition it is $\{0, f, 2f, \dots (c-1)f\}$ (or $\{f, 2f, 3f, \dots cf\}$). The generator must be the inverse[1] modulo n of the cardinality c, meaning that $cf = 1 \mod n$, almost closing the circle. For instance, a diatonic collection is generated by a chain of fifths: $n = 12, c = f = 7$ since $c \times f = 49 \equiv 1 \mod 12$.

The other ME sets are periodic. More precisely

Proposition 2. *If $c \wedge n > 1$, the ME set with c elements in $\mathbb{Z}_n$ can be constructed by repetition with period $n' = \dfrac{n}{c \wedge n}$ of a ME set with $c' = \dfrac{c}{c \wedge n}$ elements in $\mathbb{Z}_{n'}$.*

For instance, a ME Set for $n = 12, c = 8$ is $\{0, 2\} \in \mathbb{Z}_3$ repeated with period 3, i.e.

$$\{0, 2\} \oplus \{0, 3, 6, 9\} = \{0, 2, 3, 5, 6, 8, 9, 10\}.$$

Please note that this case includes the degenerate/trivial case of regular divisions of the chromatic aggregate (like a diminished seventh, say), that Ian Quinn called *type II-a* ME sets: they are a periodic repetition of a one note ME set. This case could have been treated apart since the product in the conjecture is then always nil.

The value of Fourier coefficients of any generated pc-set can be computed:

Proposition 3. *Let $A = \{0, f, 2f, \dots (c-1)f\}$ be a generated pc-set with generator f and cardinality c in $\mathbb{Z}_n$.*

Then in general $f(t) = e^{i\pi(1-c)t/n} \dfrac{\sin \frac{cft\pi}{n}}{\sin \frac{ft\pi}{n}}$.

When the denominator is nil (meaning that n is a divisor of ft) then $f(t) = c$.

3. PROOFS

Lemma 1. *The quantity $f(a)f(b)\overline{f(c)}$ is invariant by transposition.*

Proof. From Prop. 1, transposing any pc-set A by τ multiplies the product $f(a)f(b)\overline{f(c)}$ by $e^{-2i\pi\tau a/n} e^{-2i\pi\tau b/n} \overline{e^{-2i\pi\tau c/n}} = e^{-2i\pi\tau(a+b-c)/n} = 1$ since $a + b = c$. □

This allows to begin our ME set with 0, without loss of generality. Until further notice, we take $A = \{0, f, 2f, \dots (c-1)f\}$. Using Prop. 3 yields

Lemma 2. $f(a)f(b)\overline{f(c)} = e^{i(1-c)fa\pi/n} e^{i(1-c)fb\pi/n} e^{-i(1-c)fc\pi/n} \dfrac{\sin \frac{fca\pi}{n}}{\sin \frac{fa\pi}{n}} \dfrac{\sin \frac{fcb\pi}{n}}{\sin \frac{fb\pi}{n}} \dfrac{\sin \frac{fcc\pi}{n}}{\sin \frac{fc\pi}{n}}$.

When $a + b = c$, only the sines remain: $f(a)f(b)\overline{f(c)} = \dfrac{\sin \frac{fca\pi}{n}}{\sin \frac{fa\pi}{n}} \dfrac{\sin \frac{fcb\pi}{n}}{\sin \frac{fb\pi}{n}} \dfrac{\sin \frac{fcc\pi}{n}}{\sin \frac{fc\pi}{n}}$.

[1] Its opposite works just as fine.

The denominators are well defined because f, c are coprime with n, hence n cannot divie fa or fb by Gauss's Lemma. Already we see that the product of Fourier coefficients $f(a)f(b)\overline{f(a+b)}$ is real valued (this holds for any generated sequence, even when the ratio of sines appears as $0/0$).

It is a good moment to point out that if a product is real positive for a given generated pc-set, then for its complement the same product will be negative (all three coefficients being turned into their opposites).

To study the signum of this combination of sines, we are led to scrutinize the integer and fractional values of expressions like $\frac{fa}{n}$, making heavy use of

Lemma 3. *$\sin(x\pi)$ is positive when $\lfloor x \rfloor$ is even, else negative.*

For instance, for a diatonic collection, $c = f = 7, n = 12$. Choosing (say) $a = 3, b = 4$ we get the combination of sines

$$\frac{\sin\frac{7\times7\times3\pi}{12}}{\sin\frac{7\times3\pi}{12}}\frac{\sin\frac{7\times7\times4\pi}{12}}{\sin\frac{7\times4\pi}{12}}\frac{\sin\frac{7\times7\times7\pi}{12}}{\sin\frac{7\times7\pi}{12}} = \frac{\sin\frac{147\pi}{12}}{\sin\frac{21\pi}{12}}\frac{\sin\frac{196\pi}{12}}{\sin\frac{28\pi}{12}}\frac{\sin\frac{343\pi}{12}}{\sin\frac{49\pi}{12}}$$
$$= \frac{\sin(12.25\pi)}{\sin(1.75\pi)}\frac{\sin(16.33\pi)}{\sin(2.33\pi)}\frac{\sin(28.583\pi)}{\sin(4.083\pi)}$$

where from the last Lemma, only one sine ($\sin(1.75\pi)$) is negative: this proves the conjecture for that case.

Notice that omitting the factor π, all these fractions, e.g. $\dfrac{c}{n}$, belong to $G = \dfrac{1}{n}\mathbb{Z}$, a monogenous additive group containing $\mathbb{Z}$. In particular, $\dfrac{fc}{n} = \lfloor\dfrac{fc}{n}\rfloor + \dfrac{1}{n}$, exceeding the closest integer by the smallest possible value in G.

Let us denote $\dfrac{fa}{n}$ as $(fa/n)_i + (fa/n)_f$, where $(fa/n)_i$ (resp. $(fa/n)_f$) is the integer part of the fraction (resp. the fractional part). We have

$$\big((fa/n)_i + (fa/n)_f\big) + \big((fb/n)_i + (fb/n)_f\big) = \big((fc/n)_i + (fc/n)_f\big)$$

but since $\dfrac{1}{n} \leq (fa/n)_f, (fb/n)_f < 1$ and $(fc/n)_f = \dfrac{1}{n}$, we must have $(fa/n)_f + (fb/n)_f = 1 + \dfrac{1}{n}$: there is a 1 carry in the addition.

We conclude that $(fa/n)_i + (fb/n)_i + 1 = (fc/n)_i$, which compels the tally of odd integers in this identity to be 1 or 3 (one or all). For instance, if $n = 12, f = c = 5$ and (say) $a = 2, b = 3$, the fractions evaluate to $0.833, 1.25$ and their sum is 2.166: the sum of the fractional parts, 1.166, exceeds 1 by the smallest possible margin; besides, only $(fb/n)_i$ is odd.

Let us split cases:

Lemma 4. *Assume that the pc-set is* **evenly** *Maximally Even : $fc = 1 \mod 2n$. Then $f(a)f(b)\overline{f(c)}$ is real negative.*

Proof. From previous computation and the additional hypothesis, we get

$$\frac{fcx\pi}{n} = \frac{x\pi}{n} \mod 2\pi \text{ for } x = a, b, c,$$

hence $f(a)f(b)\overline{f(c)} = \dfrac{\sin\frac{a\pi}{n}}{\sin\frac{fa\pi}{n}}\dfrac{\sin\frac{b\pi}{n}}{\sin\frac{fb\pi}{n}}\dfrac{\sin\frac{c\pi}{n}}{\sin\frac{fc\pi}{n}}$. Since $a/n, b/n, c/n$ are smaller than 1, the numerators are all positive.

If either $(fa/n)_i$ or $(fb/n)_i$ is even, then the other one must be odd because their sum $(fa/n)_i + (fb/n)_i = (fc/n)_i - 1$ is odd by assumption. Hence $\sin\frac{a\pi}{n} > 0, \sin\frac{c\pi}{n} > 0$ and $\sin\frac{b\pi}{n} < 0$ (or the reverse), and the conjecture stands.

□

Lemma 5. *Assume that the pc-set is* **oddly** *Maximally Even :* $fc = n+1 \mod 2n$*, i.e.* $\dfrac{fc-1}{n}$ *is odd, meaning* $\sin\dfrac{fc\pi}{n} = -\sin\dfrac{\pi}{n}$.

Then $f(a)f(b)\overline{f(c)}$ *is again real negative.*

Proof. Now $\dfrac{fca\pi}{n} = a\pi + \dfrac{a\pi}{n} \mod 2\pi$, and similarly for the other fractions. Hence $\sin\dfrac{fca\pi}{n} = (-1)^a \sin\dfrac{a\pi}{n}$, therefore $f(a)f(b)\overline{f(c)} = (-1)^{a+b+c}\dfrac{\sin\frac{a\pi}{n}}{\sin\frac{fa\pi}{n}}\dfrac{\sin\frac{b\pi}{n}}{\sin\frac{fb\pi}{n}}\dfrac{\sin\frac{c\pi}{n}}{\sin\frac{fc\pi}{n}}$. Since $(-1)^{a+b+c} = (-1)^{2c} = +1$, we find the same formula as in the last case and a similar discussion yields the conjecture, with this time $(fa/n)_i$ and $(fb/n)_i$ both odd or both even. □

An example may be useful: $n = 15, c = 8, f = 2, c\,f = 1\times n+1, a = 3, b = 5$ yields

$$\frac{\sin\frac{a\pi}{n}}{\sin\frac{fa\pi}{n}}\frac{\sin\frac{b\pi}{n}}{\sin\frac{fb\pi}{n}}\frac{\sin\frac{c\pi}{n}}{\sin\frac{fc\pi}{n}} = \frac{\sin 0.2\pi}{\sin 0.4\pi}\frac{\sin 0.333\pi}{\sin 0.667\pi}\frac{\sin 0.533\pi}{\sin 1.067\pi}$$

with again only one negative sine in the denominator ($\sin(1.067\pi)$).

Now for the last step:

Lemma 6. *The conjecture is true for other types of ME sets.*

Proof. From Prop. 2 we consider a (type I) ME set A' with c' elements in $\mathbb{Z}_{n'}$ and repeat it with period p to build a ME set A in $\mathbb{Z}_n$: $A = A'' \oplus p\mathbb{Z}_n$ (with $A'' \subset \mathbb{Z}_n$ being a fiber of A', i.e. the same elements but lifted modulo n). Now we rely on the computation of a Fourier coefficient for a periodic pc-set:

$$\begin{aligned} f_A(t) = \sum_{k\in A} e^{-2i\pi kt/n} &= \sum_{k=k'+jp\in A} e^{-2i\pi kt/n} = \sum_{j=1}^{n'}\sum_{k'\in A''} e^{-2i\pi(k'+jp)t/n} \\ &= \sum_{j=1}^{n'} e^{-2i\pi jpt/n}\sum_{k'\in A''} e^{-2i\pi k't/n} = \sum_{j=1}^{n'}\left(e^{-2i\pi pt/n}\right)^j f_{A''}(t) \end{aligned}$$

where the sum is often 0, except when pt is a multiple of n in which case the sum is equal to n', real positive. In that case, t is a multiple of $n' = n/p$. Let $t = t'n'$, then $e^{-2i\pi k't/n} = e^{-2i\pi k't'n'/n} = e^{-2i\pi k't'/p}$ and we get $f_A(t) = n' f_{A'}(t')$. (the non nil coefficients of a periodic pc-set are equal, up to a constant, to those of the projection of the pc-set on a period. See Fig. 1 for ME Set $\{0,3,5,7\}$ modulo 9 repeated to build ME Set $\{0,3,5,7,9,12,14,16\}$ modulo 18)

We sum up: if we compute the product $f_A(a)f_A(b)\overline{f_A(c)}$, it is either (usually) 0, or $n' \times f_{A'}(a')f_{A'}(b')\overline{f_{A'}(c')}$ with $a'+b' = c'$, the cardinality of A'. Since A' is a type I ME set, we have already proved in Lemmas 4 and 5 that this is real negative, and the conjecture is thus proved in all cases.

□

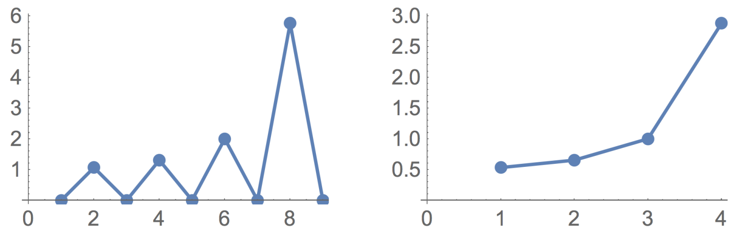

FIGURE 1. Fourier amplitudes for a motif mod. 9, right, and the same motif repeated mod. 18, left