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Iannis Xenakis's lasting legacy in mathematics of music

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Abstract: *As a composer and theorist, Yannis Xenakis has raised the level and quality of mathematics involved in music to unprecedented heights. He inspired generations of composers and also stimulated research and development in the theory of music. The present paper explores how influential some of his ideas have proved to be in numerous domains.*

Key-words: *mathematics and music, sieve, iteration, set theory, stochastic*

1. Introduction

Iannis Xenakis definitely changed the importance of mathematics in music. He proved unafraid to raise the level of the former to unprecedented heights, that could be unreachable for most listeners. However, the mathematical structure behind his works had perceptible evidence and vindicated its utilisation. The author of this paper remembers vividly the life-changing experience of *Polytopes* in the musée de Cluny, Paris, in 1972, where mysteriously the interplay of lights and notes exhibited a coherence that cried out to be elucidated.
While the concept of 'outside-time structure' can be viewed as a generalisation of previous notions, such as scales or tonalities, Xenakis used it in a conceptual revolution wherein set of pitches (for instance) are operated upon by algebraic operations, and the actual musical composition draws notes at random in the resulting sets. This paper explores four similarly ground-breaking ways of thinking, selected for the durable influence on both composition and theory: sieves, iterations, geometrical spaces, and stochastic processes.

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2. Sieves: punching holes

It was traditional in music theory to build material mostly by *adding new elements*: starting from a pitch and adding its octaves, then its fifths for instance; or glueing together tetrachords to build scales, maqams, srutis, etc. Xenakis reverses the perspective as he preserves some notes in the original, regular infinite sequence while suppressing the others. This process he called sieving.

2.1. Historical example: the sieve of Eratosthenes

This most ancient algorithm, well known to Xenakis, produces the list of primes by removing successively all composite integers. From the complete list of all integers larger than 1 (up to some range), 2 3 4 5 6 7 8 9 10…

- select 2 and scratch out all its multiples: 4, 6, 8…
- go to the next integer, 3, and remove all its multiples: 9, 15, 21… (the other ones have been scratched already)
- carry on (the next remaining integer is 5, etc).

When the end on the list is reached, all remaining numbers are prime. This simple process is still in use for producing tables of “small” primes. We will see below some of its musical inheritance.

2.2. Sieving in Xenakis compositions

The set-theoretic operations used on lists of integers, representing pitches or onsets for instance, are union, intersection, complementation, symmetrical difference. Not all remove elements, but Xenakis usually insisted on this dynamic. For instance in *Herma*[2] for piano solo (1962), he works on three sets of integers A, B, C generated by some prime multiples, plus the set R of all integers for purposes of complementation (see [Montague 1995]). Clearly there is here an aspiration to the beautiful abstraction of the boolean algebra of sets, but no less clearly (from Xenakis's own theoretical texts) the aesthetics move towards the *rarefaction* of available material, from the indiscriminate totality of R to the meaningfulness of, say, $A \Delta B$ (the subsets of elements of either A or B but not both), which can be prolonged *in-time* either by A or by B or their complements. I will refrain from listing all manifestations of these techniques, which has been done in numerous papers, and instead mention subsequent works that seem to inherit this philosophy.

2.3. Classical notions reinterpreted as sieving

Run of the mill notions such as periodic rhythms or ordinary musical scales can be presented in many different ways. I argue that looking at them as sieves — removing elements rather than adding new ones — is an important trend in contemporary theory and composition. For instance, although a diatonic scale can be

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The name of the piece, ἕρμα, refers to the idea of processing.

described as a specific sequence of semitones and tones 2212221, starting from the tonic, the generation as a *truncated chain of fifths* (say F C G D A E B for C major) has gained influence, and is a specific instance of the more general notion of Maximally Even Set, developed by Jack Douthett [Clough & Douthett 1990] with several different co-authors. The simplest way to create all ME sets of d notes in a c element cyclic universe uses the almost arithmetic sequence of values[3]

$$J_{c,d,\alpha}(k) = \lfloor k\frac{c}{d} + \alpha \rfloor \mod c, \quad k = 0..d-1$$

which *truncates* the last terms of the sequence (from k=d to c). For (c, d)=(12, 7) one finds all diatonic scales, changing d to 5 would yield their complements, the pentatonic scales; for c = 8 and d = 3 or 5 one finds the ubiquitous *tresillo and cinquillo* rhythms, etc. (The complement of a ME set is always a ME set.)

Yet another process for generating (some) ME sets, even more reminiscent of Eratosthenes, generalizes the sequence of fifths for the diatonic, by iterating the interval f defined as the multiplicative inverse of d modulo c and truncating the sequence to d elements. For instance in the diatonic case, f is equal to 7 like d, since

$$f \times d = 7 \times 7 = 49 = 1 + (4 \times 12) \equiv 1 \mod 12$$

Although this definition is equivalent to the previous one, it involves more advanced maths, i.e. group theory. Identifying ME sets and comparing other sets to them has become a mainstay of XXI*st* century music theory, they being involved in Gottfried Toussaint's Euclidean Rhythms [Gómez-Martín 2009] as well as measuring the octatonicity of slavic music at the turn of XIX*th* century [Amiot 2017].

2.4. Rhythmic canons

Another innocent musical object has been involved in formidable mathematical problems and conjectures. Although as far as I know Xenakis did not write any, rhythmic canons are definitely instances of his philosophy: the outside-time structure is an algebraic equation between discrete sets:

$$A \oplus B = \mathbb{Z}/n\mathbb{Z}$$

which means in musical terms that the motif A, copied at different offsets listed in B, fills in the time line exactly: A corresponds precisely to what it carves out. The following instance is a very short example (actually a birthday card) of a rhythmic canon composed by Georges Bloch.

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[3] The ratio is not an integer in general, but the values are rounded to the largest integer smaller than them. For instance 3x12/7 = 5.142… is replaced by 5.

Fig. 1. *A birthday card rhythmic canon*

Measuring in sixteenth notes and beginning with 0, motif *A* is simply 0 4 5 9 (see bass voice in first bar). It is offset 4 times, respectively by 0, 6, 8 and 14 which together constitute *B*. Each version is repeated indefinitely every 16 notes (e.g. *A*+8 gives 8 12 13 **1** since 9+8 = 1 modulo 16, cf. the middle upper voice in the repeated bar).

Apparently simple queries, such as determining all possible canons (for a given length) or even testing easily whether a given motif A can produce a canon, are still open questions and raise many deep conjectures. The study of rhythmic canons originates in the huge solo work of D.T. Vuza [Vuza 1991-93], who rediscovered a bevy of results of commutative algebra originating in a conjecture stated by Hajòs in 1948. It is one of these areas where musical ideas enabled progress in so-called 'pure' mathematics, see [Amiot 2011]. One disproved conjecture stated that in a canon, either *A* or *B* should be a palindrom, just as the motives in the next example which raises the level of maths in the inheritance of Xenakis to dizzying heights.

2.4. From Eratosthenes to Riemann's hypothesis

Perhaps the greatest unsolved mathematical conjecture is the Riemann hypothesis, which states that all roots of the ζ function lie on a given line in the complex plane (Re $z = 1/2$). Whereas the metaphor of considering these roots as musical pitches has been advanced, Field medalist Alain Connes [Connes 2021] went further and discovered interesting discrete rhythmic structures in complicated mathematical objects which (typically) generalize the unsolved problem. To quote him,

> *To an hyperelliptic curve of genus g corresponds for each prime number p not dividing the discriminant a collection of 2g time onsets with profound mathematical meaning which repeat in a periodic manner with a frequency log p.*

More practically, lists of such onsets appear on the next figure.

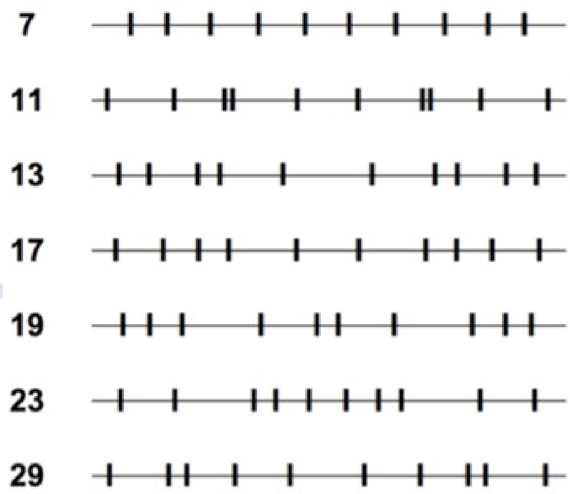

Fig. 2. *Rhythms obtained from hyperelliptic curves on finite fields*

These palindromic motives are parametrized by prime numbers, neatly tying the loop of this section with Eratosthene's sieve, which Connes used in a choreography putting in-time the above rhythms as can be seen following this link: https://www.dropbox.com/s/ld1z64em5u3mxjx/fullvideo.mp4?dl=0

3. Iterations

Eratosthene's sieve is actually an *algorithm*: the same procedure is applied again and again until the final result is obtained. The spirit of exhausting the initial material is frequent in Xenakis practice — keeping in line with the example of *Herma*, he computed the total number of sets that could be obtained from *A, B, C,* and *R* with boolean operations. The essence of this systematic mind[4] (which in itself inspired *combinatorial composers*) appears to be the process of iteration, which was used in a very abstract context in the composition of *Nomos Alpha* for solo Cello. The operation which is repeated again and again is the composition of the two last elements obtained. In a simpler context, this is the definition of the Fibonacci sequence:

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4 "J'ai pu aller plus loin dans la compréhension interne de la musique, mais aussi dans sa pratique, *en recherchant toutes les possibilités mathématiques* des combinaisons sonores que j'inventais", in *Le fait culturel,* 1980. I stressed "*looking for every mathematical possibility*".

1, 1, 1+1=2, 2+1=3, 3+2=5, 5+3=8, …

although Xenakis did it not with numbers but with ùuch more complicated objects, the rotations of a cube (perhaps inspired by his architectual persona), which lie in a mathematical structure called a group. This group has 24 elements, which means that any Fibonacci-like sequence[5] must repeat itself since there is a limited, finite number of possible consecutive pairs:

... ***a b*** [*b.a (b.a).b* ... ...] ***a b*** [*b.a* ...] ***a b*** [...]

When one pair *a b* is found again, as it must, the sequence will repeat itself exactly from this point, and indefinitely. There is, or was, no general study of such sequences in a group, but Xenakis found and used only maximal cases, which happen to loop after 18 iterations and involve 13 distinct elements out of 24 [Andreatta 2012]. In the talk, instead of the actual moves on the cube drawn by Xenakis (not unlike Rubik's cube! See [Mannone 2022] for a recent musical application), I showed the sequence of rotations using their 3x3 matrixes, alluding to the Markovian processes evoked in the last section. Xenakis himself used pictures of the rotating cube, pinpointing the permutations of its vertices which were the actual musical cells to be played by the cellist.

3.1. Vieru sequences

Slightly later, Anatol Vieru also raised the ante by using iterations in a highly abstract context. Considering sequences of integers modulo some number (usually pitch-classes, modulo 12), he iterated the shift and difference operators. For instance, starting with the sequence {2, 3, 5, 7}, its right shift is {3, 5, 7, 2} and the difference is {3, 5, 7, 2} - {2, 3, 5, 7} = {1, 2, 2, 7} (computing modulo 12). Applying the same operation to the result, we find {1, 0, 5, 6}. After a few rounds we reach {8, 0, 4, 0}, and this initiates a periodic sequence with period 8.

Vieru used this technique to produce numerous twelve-tone rows as compositional material. It can be compared with the *séries proliférantes* previously used by Jean Barraqué (for instance in his *Sonate*) who composed a row with itself considering it as a permutation of 12 elements. But Vieru's sequences can be much longer.[6] As he had noticed, the behaviour exemplified above is general [Andreatta 2004]: after some time, the sequence gets periodic (for reasons similar to the preceding case).

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5 Indeed any recursive sequence, defined by applying the same operation on instances of the sequence.

6 (3, 2, 0, 6, 7, 1, 1) has period 22,568 modulo 15. A *série proliférante* cannot produce more than 60 tone rows.

This is well known in general algebra:[7] iteration of any linear process can be decomposed as a *nilpotent* part, which annihilates the transitory element of the initial signal after a finite number of iterations, and a *permutational* part, which rotates the periodic component of the signal in the finite case.

3.2. Johnson's autosimilar melodies

A last example of extracting a significant structure by iteration is provided by Tom Johnson's *autosimilar melodies*, such as the one in Fig. 3.

Fig. 3. *Extracting one note out of five yields the same melody*

Such a melody can be defined as self-invariant under a sieving operation.[8] As mentioned in [Amiot 2008], they also appear as fixed points of the iteration of such an extraction applied to just any initial periodic melody.

Systematically and even doggedly repeating the same computation allows structure and meaning to emerge from chaos, just like the sculptor chipping at a raw stone eventually produces his *chef d'œuvre*.

4. Geometrical spaces in music

The philosophy of stating an outside-time structure, or more generally a space of possible states/possibilities, can be related to geometry: instead of focusing on points (notes), the composer adopts a larger perspective, considering their sets, and their properties and relations, just as Euclide would consider triangles, circles, squares and how they intersect. This was of course reflected in the building and use of the UPIC machine, whose graphical user interface was ahead of time — consider that

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7 The most general statement is known as Fitting Lemma, see any textbook on commutative algebra.

8 Johnson called them *selfsimilar*. Historical examples abound, and can be found for instance in Scarlatti, Beethoven, Glen Miller…

computer mouses were only produced by Apple in 1986 — perhaps because it was inspired by musical thinking, not desktop productivity. It is illuminating to compare with the X4, developed contemporarily in IRCAM, which was much more focused on sound events and generation than on geometric thinking.
There is no doubt that the modern spirit of defining musical spaces (vector spaces, modules, cyclic groups) and singling out meaningful subspaces and their interactions, or operations on them, is inspired by Xenakis (see Fig. 4 for an ordinary example taken from a pregraduate class). This philosophy permeates all modern music theory, including all of the American school, with Babbitt, Forte, Lewin, Morris, Rahn [Rahn 1980], developing the deceivingly dubbed "Set Theory" which is more about groups of operations on pitch-class sets or rhythms[9]; there is also a revival of graph theory with Euler's *Tonnetz* [Cohn] and generalizations to other *tonnetze* [Bigo 2013] or to similar torii [Amiot 2013]; much tougher mathematics involve homology and topological algebra with Mazzola and his Topos [Mazzola 2018], gestures and hypergestures [Mazzola 2020], and the quotient homogeneous spaces like Callender, Quinn and Tymoczko's orbifolds [Callender 2007] which model sets of pitches modulo diverse groups of operations like parts of continuous vector spaces twisted and folded.

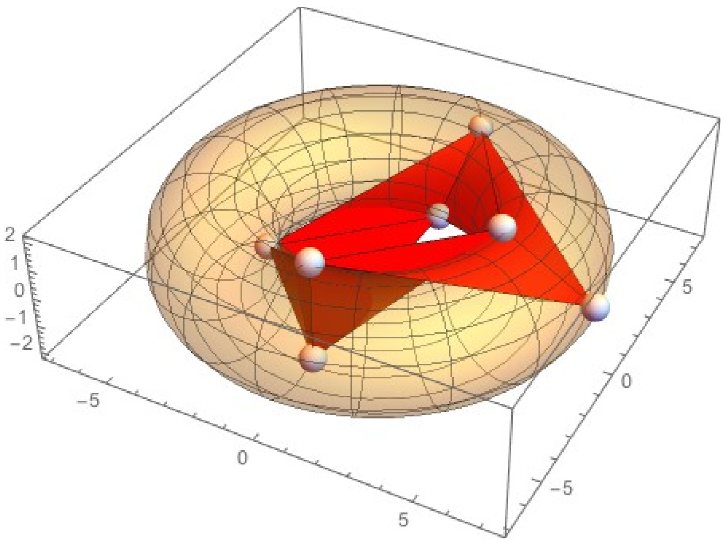

Fig. 4. *The 7 C major triads form a Möbius strip on the torus of thirds.*

5. Stochastic processes

9 Including the group T/I of transpositions and inversions, and the permutation group.

After defining an outer-time structure by set operations, Xenakis often chose elements in it with a random process. I will stress the essential differences between this computerized process, where non-homogeneous distributions are used and modelized, and John Cage's use of Yi King sticks in *Numbers* and numerous other pieces, with uniform law (any result is just as good as any other) and human, by-hand sampling. In the former case, there is a thoughtful structuration of the material and extraction process. In the latter, *Tout est art,* meaning anything goes. Perhaps even worse were the Minuets and Waltzes obtainable at the throw of a pair of dice and composed by Mozart or Haydn, which were intended as pleasantries.
Producing stochastic music was very much *en vogue* in the 70's and 80's and has perhaps lost momentum afterwards. However, the idea that mere chance was capable of modeling meaningful structures has pervaded our culture to an unpredictable level. For instance, Xenakis explicitly used the principle of a Markovian process, wherein the same transition probability matrix[10] is applied repetitively — another nice example of iteration. With greatly improved capacities in data mining, our century has developed *inverse* Markovian analysis, where the probability matrix is inferred from the data (in music, see for instance [Shapiro 2021] with references). This is how our mobile phones predict the text we are typing, or, more relevant to music, a way to check for authentication of works of art, and (allegedly) how pop hits are written. I include an example derived from Big Ben's chime of a Markovian automaton and its transition probabilities. It is caricatural but when it was first implemented in 1984 it was very much inspired by Xenakis's spirit. The arrows and figures mean that, for instance on reaching A there is an equal chance that the next note will be B or D.

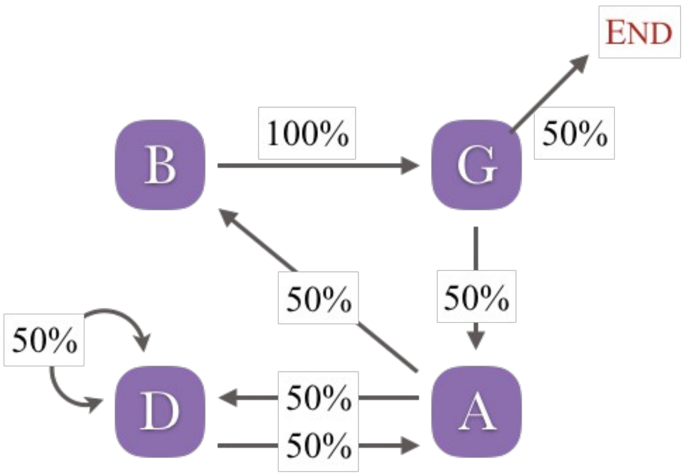

Fig. 4. *Example of a Markovian process.*

10 It is called a *stochastic matrix*. In the example on Fig. 4, the entries take values 0, 1 or ½.

These probabilities provide a good chance (3%) to recreate the original chimes, together with any number of variants.

6. Conclusion

Xenakis dared to give mathematics an explicit and prominent place in his musical creation — true mathematics, not Kindergarten stuff. We have perhaps forgotten how shocking it appeared at the time. But the musical world was henceforth changed for ever, with consequences that Xenakis could not have foreseen: although some of the concepts that he made use of may seem outdated nowadays, even more abstract and cutting-edge mathematics tools are used in music as a matter of course, and shape our vision of how music is made and perceived. The creation of the thriving *Society for Mathematics and Computation in Music* in 2007 testifies to that.

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