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# David Lewin and Maximally Even Sets

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David Lewin originated an impressive number of new ideas in musical formalized analysis. This paper formally proves and expands one of the numerous innovative ideas issued by Ian Quinn in his dissertation [17], to the import that Lewin might have invented the much later notion of *Maximally Even Sets* with but a small extension of his very first published idea, where he made use of Discrete Fourier Transform (DFT) for investigating the intervallic differences between two pc-sets. Many aspects of Maximally Even Sets (ME sets) and, more generally, of generated scales, appear obvious from this original starting point, which would deserve in our opinion to become standard. In order to vindicate this opinion, we develop a complete classification of ME sets starting from this new definition. As a pleasant by-product we mention a neat proof of the hexachord theorem, which might have been the motivation for Lewin's use of DFT in pc-sets in the first place. The nice inclusion property between a ME set and its complement (up to translation) is also developed, as it occurs in actual music.

Keywords: Maximally Even Sets, Discrete Fourier Transform, David Lewin.

*Notations : the cyclic group of order c is $\mathbb{Z}_c$. It models a chromatic universe with c pitch classes, and it is as usual pictured as a regular polygon on the unit circle. In most actual examples c will be equal to 12.*
*$x \mid y$ means the integer $x$ divides $y$.*
*For the sake of readability we generally use the same notation for integers and their residue classes, the context usually making clear whether a computation occurs in $\mathbb{Z}$ or in $\mathbb{Z}_c$.*
*The greatest common divisor of $x, y$ is denoted by $\gcd(x, y)$.*
*We will use indiscriminately 'Fourier transform', 'Discrete Fourier Transform', or 'DFT'.*
*The bracket notation is for the floor function.*
*The symbol $X \oplus Y$ means 'all possible sums of an element of $X$ and an element of $Y$', each result being obtained in a unique way.*

## 1 Fourier Transform of pc-sets

Part of our claim that Fourier Transforms provide the best way to define Maximally Even Sets relies on the high musical significance of the DFT of pc-sets in general. This was salient in [17] for the special pc-sets that Quinn collected as 'prototypes', among which the ME sets; and it was confirmed since by many other cases. We thus feel it important to spend some time on the general DFT of pc-sets before turning to the main topic, that is its application to ME sets proper.

### 1.1 *History*

In a short paper ( [13]), D. Lewin investigated intervallic relationships between two 'note collections' and proved that, except in several listed exceptional cases, the interval function between the 'note collections' enables to reconstruct one from the other. He cursorily motivates the five exceptional cases by a final note, wherein he puts forward that

(1) the interval function is a convolution product (of characteristic functions),
(2) the Fourier transform of such a product is the ordinary product of Fourier transforms.

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This shows that (when the Fourier transform of the characteristic function of $A$ is non zero) knowledge of $A$ and of the interval function yields complete knowledge of the characteristic function of $B$.

In current modern notations, defining the interval function between $A, B \subset \mathbb{Z}_c$ as

$$IFunc(A,B)(t) = \mathrm{Card}\{(a,b) \in A \times B, a - b = t\},$$

the characteristic fuction of $X$ as $1_X(t) = \begin{cases} 1 & \text{if } t \in X \\ 0 & \text{if } t \notin X \end{cases}$, $IFunc$ appears immediately as the convolution product of the characteristic functions of $A$ and $-B$:

$$1_A \star 1_{-B} : t \mapsto \sum_{k \in \mathbb{Z}_c} 1_A(k) 1_{-B}(t-k) = \sum_{k \in \mathbb{Z}_c} 1_A(k) 1_B(k-t) = IFunc(A,B)(t)$$

as $1_A(k)1_B(k-t)$ is nil except when $k \in A$ and $k - t \in B$. Hence from the general formula for the Fourier transform of a convolution product,

$$\mathcal{F}(IFunc(A,B)) = \mathcal{F}(1_A) \times \mathcal{F}(1_{-B})$$

where $\mathcal{F}(f)$ stands for the discrete Fourier transform of a map $f$.

We will not quote the formula given by Lewin himself, as it is hardly understandable: his notations are undefined and the computations extremely cursory. Of course this is not for lack of rigor: as the following quotation suggests, Lewin did not really hope to be understood when talking mathematics.

> The mathematical reasoning by which I arrived at this result is not communicable to a reader who does not have considerable mathematical training. For those who have such a training, I append a sketch of the proof : consider the group algebra [...] [13]

Reading Lewin's paper gives one a strong feeling that he wrote as little as possible on the mathematical tools that underlay his results. Indeed, what little he mentioned did rouse some readers to righteous ire in the following issues of JMT.

Nowadays such a 'considerable mathematical training' will be considered basic by many readers of this journal; for instance D.T. Vuza made use of the equation above in the 80's in the course of his monumental research about rhythmic canons (see [21]), wherein he stressed the importance of Lewin's use of DFT of characteristic functions.

And as we will endeavour to prove, this approach enables to define ME sets (in equal temperament) in a way perhaps more suggestive and even intuitive, than historical/usual definitions.

## 1.2 *A quick summary of Fourier transforms of subsets of $\mathbb{Z}_c$*

### 1.2.1 *First moves.*

Definition 1.1 *Following Lewin, we will define the Fourier transform (or DFT in short) of a pc-set $A \in \mathbb{Z}_c$ as the Fourier transform of its characteristic function $1_A$:*

$$\mathcal{F}_A = \mathcal{F}(1_A) : t \mapsto \sum_{k \in A} e^{-2i\pi kt/c}$$

*The values $\mathcal{F}_A(t), t \in \mathbb{Z}_c$, are the Fourier coefficients.*

$1_A$ is a map from $\mathbb{Z}_c$ to $\mathbb{C}$, whose DFT is well defined for $t \mod c$, as $\mathcal{F}_A(t+c) = \mathcal{F}_A(t)$[1] unless otherwise indicated.

[1] When the choice of a representative in a residue class is relevant, we will specify it; otherwise the presence of an integer in a computation in $\mathbb{Z}_c$ will mean any of its representatives.

The DFT of a single note $a$ is a single exponential function $t \mapsto e^{-2i\pi at/c}$, the DFT of the whole chromatic scale is

$$\mathcal{F}_{\mathbb{Z}_c}(t) = \sum_{k=0}^{c-1} e^{-2i\pi kt/c} = 0 \qquad \text{for all } t \in \mathbb{Z}_c \text{ except } t = 0.$$

But $\mathcal{F}_A + \mathcal{F}_{\mathbb{Z}_c \setminus A} = \mathcal{F}_{\mathbb{Z}_c}$, hence

LEMMA 1.2 *The Fourier transforms of a pc-set $A$ and of its complement $\mathbb{Z}_c \setminus A$ have opposite values, except when $t = 0$:*

$$\forall t \in \mathbb{Z}_c, t \neq 0, \qquad \mathcal{F}_{\mathbb{Z}_c \setminus A}(t) = -\mathcal{F}_A(t)$$

Furthermore, we get $\mathcal{F}_{\mathbb{Z}_c \setminus A}(0) = \mathcal{F}_A(0)$ if and only if Card $A = c/2$, as

LEMMA 1.3 *The Fourier transform of $A$ in 0 is equal to the cardinality of $A$: $\mathcal{F}_A(0) = \text{Card } A$.*

The DFT $\mathcal{F}_A$ characterizes the pc-set $A$, by the following identity (Inverse Fourier transform)

$$1_A(t) = \frac{1}{c} \sum_{k \in \mathbb{Z}_c} e^{+2ik\,t\pi/c} \mathcal{F}_A(k)$$

easily derived from the definition of $\mathcal{F}_A$. Thus the DFT yields the same information as the pc-set, but *in a form that stresses musically relevant concepts*. More precisely, there is preservation of the absolute value of DFT under all usual[1] musical transformations. For instance,

THEOREM 1.4 *The* length $|\mathcal{F}_A|$ *of the Fourier transform is invariant by (musical) transposition or inversion of the pc-set $A$. More precisely, for any $p, t \in \mathbb{Z}_c$*

- $\mathcal{F}_{A+p}(t) = e^{-2ip\pi t/c} \mathcal{F}_A(t)$ *(invariance under transposition)*
- $\mathcal{F}_{-A}(t) = \overline{\mathcal{F}_A(t)}$ *(invariance under inversion)*

Thus $|\mathcal{F}_A|$ is an invariant under the T/I group of musical transformations, and also under complementation (except in 0 when Card $A \neq c/2$). As we will see momentarily, it is not a characteristic invariant (meaning $|\mathcal{F}_B|$ may be equal to $|\mathcal{F}_A|$ though $A$ and $B$ are not T/I related) because of the famous Z-relation. All the same, it appears to be a very good snapshot of the relevant musical information of a given pc-set: by dropping the information of the *phase* of the Fourier coefficients and retaining only the *absolute value*, we seem to keep the best part, in a way reminiscent of the Helmoltzian approach of sound, which showed that the phase of a sine wave can (in many cases) be neglected, as the frequency is the part that generates the perception of pitch. This strongly vindicates and to some measure extends Quinn's ( [17]) notion of 'chord quality', which appears in the last section of his dissertation with a value that is precisely $|\mathcal{F}_A(d)|, d = \text{Card } A$, and is measured, quite appropriately, in 'lewins'.

As as nice application of these invariance properties, we may characterize periodic subsets:

PROPOSITION 1.5 *$A \subset \mathbb{Z}_c$ is periodic , meaning $A + \tau = A$ for some $\tau$, if and only if $\mathcal{F}_A(t) = 0$ except when $t$ belongs to some subgroup of $\mathbb{Z}_c$.*

*Proof* From Thm. 1.4 we have

$$\forall t \in \mathbb{Z}_c \quad \mathcal{F}_A(t) = e^{-2i\pi\tau\, t/c} \mathcal{F}_A(t) \iff \forall t \in \mathbb{Z}_c \quad (1 - e^{-2i\pi\tau\, t/c}) \mathcal{F}_A(t) = 0$$

[1] Less usual transformations, like $t \mapsto 7t \mod 12$, *permute* the Fourier coefficients. This is the meaning of the relationship between chromatic clusters and ME sets with $\gcd(c, d) = 1$ that will come to light later.

Unless $e^{-2i\pi\tau\, t/c} = 1$, this compels $\mathcal{F}_A(t)$ to be 0. Now the condition $e^{-2i\pi\tau\, t/c} = 1$ is equivalent to $c \mid \tau t$, i.e. $t$ multiple of $m = c/\gcd(c,\tau)$ – this makes sense for any representative of the residue classes $\tau$ and $t$. This is compatible with reduction modulo $c$, and means $t \in m\,\mathbb{Z}_c \subset \mathbb{Z}_c$.

Conversely, if $\mathcal{F}_A$ is nil except on a subgroup, say $m\mathbb{Z}_c$ with $0 < m \mid c$ in $\mathbb{Z}$ (we recall all subgroups of $\mathbb{Z}_c$ are cyclic) then, by inverse Fourier Transform

$$\forall k \in \mathbb{Z}_c \qquad 1_A(k) = \frac{1}{c}\sum_{t\in\mathbb{Z}_c} \mathcal{F}_A(t) e^{2i\pi\, k\, t/c} = \frac{1}{c}\sum_{t'\in m\mathbb{Z}_c} \mathcal{F}_A(t') e^{2i\pi\, k\, t'/c} = \frac{1}{c}\sum_{t''=1\ldots\frac{c}{m}} \mathcal{F}_A(m\, t'') e^{2i\pi\, k\, t''\, m/c}$$

and this is obviously periodic with (the residue class of ) $\dfrac{c}{m}$ as a period, as each term in the sum is $\dfrac{c}{m}$ periodic. □

*Remark 1*

- Some may well claim this proposition is obvious: a subset $A \in \mathbb{Z}_c$ is the set of residues of a periodic set $\widehat{A} \subset \mathbb{Z}$, with period $c$. This periodicity means precisely that $1_A$ (or $1_{\widehat{A}}$, with the same formula) can be expressed as a combination of $c$ exponential functions, the $t \mapsto e^{2i\pi\, k\, t/c}$: this is the inverse Fourier transform formula and the very reason Fourier transform works. The existence of a smaller period $m \mid c$ means that $m$ exponentials functions only are sufficient, e.g the $t \mapsto e^{2i\pi\, k\, t/m}$. To give an example in a more mundane context, the function $|\sin x|$ has a Fourier Series expansion of the form $\sum a_{2n}\cos(2n\, x)$, because the map is $\pi-$periodic, not only $2\pi$-periodic.
- It is noteworthy that the multiples of $\tau \in \mathbb{Z}_c$ appear as a subgroup $m\,\mathbb{Z}_c$ of $\mathbb{Z}_c$, where the non-negative integer $m$ is usually smaller than $\tau$ (taking $\tau$ in $[0, c[$): $m = \gcd(c,\tau)$ as we will recall in lemma 3.8 later.
- Let us define a *regular polygon* in $\mathbb{Z}_c$ as any translate of a cyclic subgroup $m\,\mathbb{Z}_c$, i.e. a set $a + m\,\mathbb{Z}_c$. These are the orbits of the translations $t \mapsto t+m$ (identifying $m$ with its residue class), and any periodic subset $A \subset \mathbb{Z}_c$ must hence be a reunion of such regular polygons.
- All of this is only interesting when $1 < m < c$.
- In $\mathbb{Z}_{12}$, the octatonic scale (0 1 3 4 6 7 9 10) is an interesting example of such a periodic subset. Its group of periods is $3\,\mathbb{Z}_{12}$. Periodic subsets of $\mathbb{Z}_{12}$ are well known as Messiaen's *Modes à Transposition Limitées.*

**1.2.2 *DFT and intervallic content.*** A word of warning is necessary here: in order to stay into the space wherein we are taking the Fourier transforms, we must consider *oriented intervals* and not the more customary notion of interval contents. Using standard notations, we consider

Definition 1.6 *The interval content of a subset $A \in \mathbb{Z}_c$ is*

$$IC_A(k) = IFunc(A, A)(k) = \text{Card}\{(i,j) \in A^2,\ i - j = k\}$$

Theorem 1.7 (Lewin's Lemma)
*The DFT of the intervallic content is equal to the square of the length of the DFT of the set:*

$$\mathcal{F}(IC_A) = |F_A|^2$$

*Proof* Let $A$ be a pc-set; as Lewin observed (for the more general interval function between two subsets), the 'intervallic function' from pc-set $A$ to itself is [1] the convolution product

$$IC_A = 1_A \star 1_{-A}$$

But as we recalled earlier, the Fourier transform of a convolution product is the ordinary product of Fourier transforms, i.e. (using last part of theorem 1.4)

$$\mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = \mathcal{F}_A \times \overline{\mathcal{F}_A} = |F_A|^2$$

□

Note that the Fourier transform of any IC is a real positive valued function, an uncommon occurence among DFT of integer-valued functions [1]. Perhaps this would be a good way to look at the vexing question of the Z-relation, which can now be reformulated in DFT term:

Definition 1.8 *$A, B \subset \mathbb{Z}_c$ are Z-related if and only if they share the same interval content; or equivalently if the absolute values of their DFT are equal: $A \mathcal{Z} B \iff |\mathcal{F}_A| = |\mathcal{F}_B|$.*

The equivalence stands because $|\mathcal{F}_A|$ holds all the information about $IC_A$ by inverse Fourier transform. Please note that we endeavour here to define a true equivalence relation, contrarily to the Fortean tradition which excludes the 'easy case', when $A, B$ are T/I related [2] – this case follows directly here from theorem 1.4.

From there we also get a very short proof of the hexachord theorem, considered by some the first mathematically interesting result in music theory.

At the time he issued his first paper, Lewin had come to work with Milton Babbitt, who was trying to prove the hexachord theorem:

Theorem 1.9 *If two hexachords (i.e. 6 notes subsets of $\mathbb{Z}_{12}$) are complementary pc-sets in $\mathbb{Z}_{12}$, then they have the same intervallic content (same numbers of same intervals).*

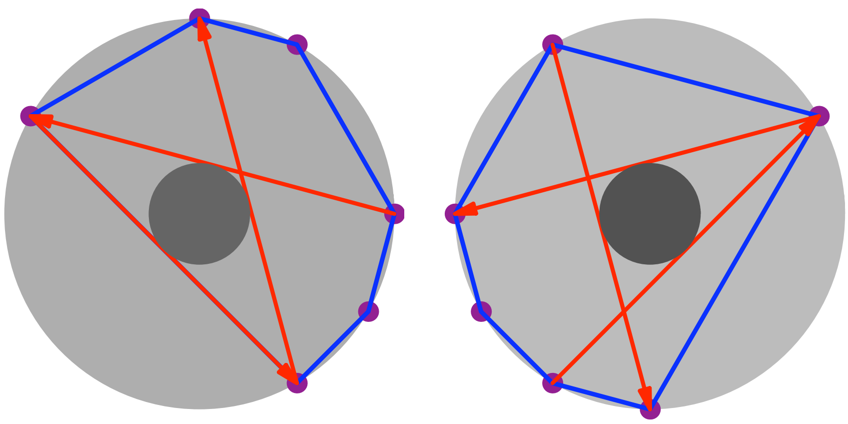

Figure 1. These two hexachords share intervallic content

On the figure 1 with two complementary hexachords, the fifths have been signaled with arrows. Each hexachord has the same number of fifths, three in this example.

[1]This relation has been quoted, in musical context, by several authors: for [21], it is the most important single contribution by David Lewin; it also appears for instance in the recent [16].

[1]The DFT of a real valued function is non real in general, it only verifies $\mathcal{F}(f)(-t) = \overline{\mathcal{F}(f)(t)}$.

[2]The traditional position is not tenable; another argument against it is that some classes of 'Z-related' chords are indeed exchanged through action of a larger group than T/I, like the two famous all-intervals (0 1 4 6) and (0 1 3 6) in $\mathbb{Z}_{12}$, which are affine-related – and this is a general situation, as any affine transform of an all-interval set will be Z-related. Jon Wild pointed out to us that the reverse is false.

A simple derivation of this theorem in $\mathbb{Z}_c$ for any even $c$ ensues from the simple properties of DFT listed already:

*Proof* If $A \in \mathbb{Z}_c$ has $c/2$ elements, then as mentioned above, $\mathcal{F}_{\mathbb{Z}_c \setminus A} = -\mathcal{F}_A$. So

$$\mathcal{F}(IC_A) = |F_A|^2 = |F_{\mathbb{Z}_c \setminus A}|^2 = \mathcal{F}(IC_{\mathbb{Z}_c \setminus A}) \qquad \text{Hence (by inverse DFT)} \quad IC_A = IC_{\mathbb{Z}_c \setminus A}.$$

□

In our opinion it is very difficult to believe that Lewin should not have been aware of this elegant proof (as far as we know, first published in [?]). We suggest that he did not produce it in 1959/60 because the maths looked too involved at the time, and did not publish it later for reasons of his own (it could be because other, more elementary proofs had appeared in the meantime). It is left to the reader, as a good, healthy and entertaining exercise, to prove in the same way the Generalized Hexachord Theorem, as presented in [18], [20], [16] among many others.

## 2 ME sets

### 2.1 *Some definitions*

#### 2.1.1 *Informal approach of ME sets.* :

Maximally Even Sets, or ME sets in short, were defined in [8], generalized in [7] and later extended to Well Formed Scales, which exist also in non equal temperaments (see [6]). The name refers to the intuitive feature of being 'as evenly distributed in the chromatic circle as possible'. As we will see, it is not so easy to make this idea rigorous: many different though equivalent definitions exist, and our main objective in this paper is to ground firmly the notion of ME sets on a DFT-based definition. We include a short paragraph for readers who might still be unfamiliar with the notion, followed by a discussion of several existing definitions. A very thorough paper on state-of-the-art applications of ME sets is [10].

Originally, Clough, Myerson and soon after Douthett observed this yet informal notion of 'maximal evenness' in a collection of famous scales: whole tone scale, major scale, pentatonic, octatonic… For musicological reasons, and perhaps also because of mathematical difficulties we will mention below, their definition was rather indirect.

In the minor scale there are three different values of intervals between consecutive notes. Not so for the major scale, or the melodic (ascending) minor; but the latter features three different fifths.

From these examples, and others, ME sets were defined in regard with the different (some say 'diatonic') possible values of intervals inside the scale: for instance, the major scale and the pentatonic alike have only two different interval sizes between consecutive notes – tones and semi-tones for the one, tones and minor thirds for the other. Also notice that the two semi-tones in the major scale, for instance, are as far from one another as possible. This has some relevance to the organisation of black and white keys on a keyboard, and hence to traditional musical notation in staves.

The common original definition (here reworded) states that

Definition 2.1 *Let $A$ be a subset of $\mathbb{Z}_c$. Let us for convenience's sake call a 'second' any interval between two adjacent elements of $A$, a 'third' an interval between every odd note, and so on.*

*Then $A$ is maximally even if, and only if, there are at most two different kinds of 'seconds', 'thirds', 'fourths' aso.*

This definition suffers from the common blemish of many formalized musicological definitions, that take for granted many notions with intuitive, musical support (like diatonic intervals, adjacency of notes, etc.) which are not so obvious to define mathematically[1].

[1]To be fair, pre-Hilbert mathematics (and some post-Hilbert, too) often relied too heavily on intuitions of the physical world, as the quarrel on non-euclidean geometries made clear.

To state it with numbers: if an *ordered* scale[2] is $A = \{a_1, a_2 \dots a_d\}$ with indexes taken modulo $d$ and values taken modulo $c$, for each value of $k$ there should be at most two different values of $a_{i+k} - a_i$ when $i$ varies.

This was named the 'Myhill property' in [8][1]) and it is not at all straightforward.

Worse still, in our opinion, this definition necessitates an ordering, or reordering, of the notes : (C E D G A) is not a ME set, though (C D E G A) is ! This verges on the unsatisfactory, if one is interested in pc-sets and not (ordered) scales.

Many geometrical criteria have been given, and proved equivalent (see [10]); we especially like the 'black and white' definition in [7], very intuitive though hardly practical (see figure 2): plot two regular polygons, one white with $d$ vertexes and one black with $c - d$ vertexes. Then rearrange all the vertexes, preserving order, with identical distance between consecutive points. Both black and white subsets are ME sets.

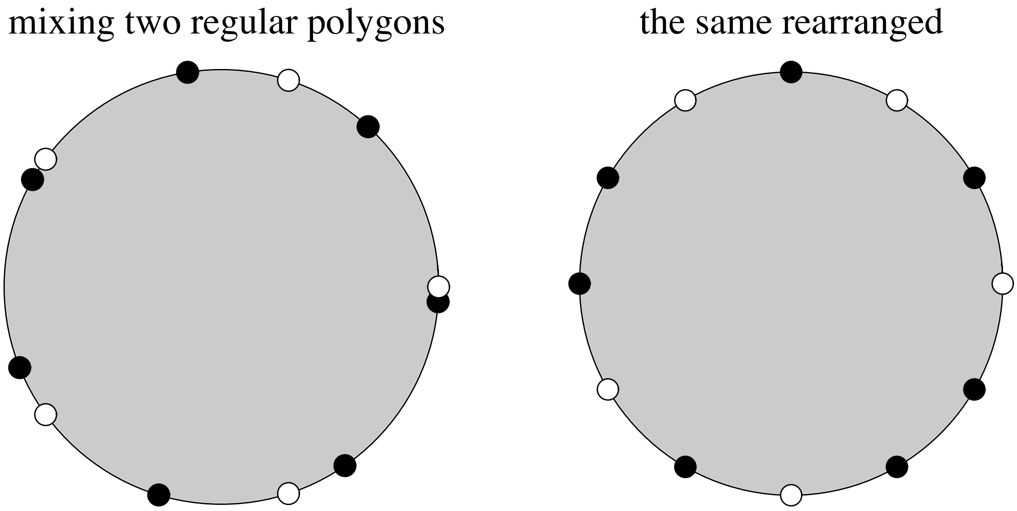


Figure 2. Respacing the points of two intertwined regular polygons

The most effective way to actually compute ME sets is as follows: taking $c$ as the cardinality of the ambient chromatic space, $d$ the number of notes of the looked-for set, and $\alpha$ some arbitrary number, the $J$ functions

$$J^{\alpha}_{c,d} : k \mapsto \lfloor \frac{kc + \alpha}{d} \rfloor, k = 0 \dots d - 1$$

already introduced in [8], give all ME sets with cardinality $d$ by their sets of values

$$J^{\alpha}_{c,d}(0), J^{\alpha}_{c,d}(1), \dots J^{\alpha}_{c,d}(d - 1)$$

(taken modulo $c$): for instance with $c = 12, d = 5, \alpha = 12$ one gets the pentatonic (0 2 4 7 9); but relevance to the intuitive idea of maximum evenness, or even to sizes of intervals, is less than obvious.

The most natural definition might be to try and maximize the mutual distances between all the notes, eg $\sum_{a,a' \in A} \delta(a, a')$, but the result depends on the chosen distance function $\delta$, and is not satisfactory for the (arguably) most natural one, the interval metric :

$$\delta(u, v) = \min_{k \in \mathbb{Z}} |u - v + kc|$$

[2]We skip a formal definition of 'ordered' in $\mathbb{Z}_c$, which will be useless in our approach.

[1]Note that in general, it is not enough that Myhill property holds for adjacent notes, e.g. having only two kinds of 'seconds' does not ensure we have a ME set, as shown by the example of the melodic minor scale.

as several unexpected[2] extraneous solutions crop up, as in figure 3 . A ‘good’ definition would be expected to give one characteristic shape for a given pair $(c, d)$, not so many. This exemplifies why there is no universal, or obvious, definition for the naïve concept of ‘Evenness’.

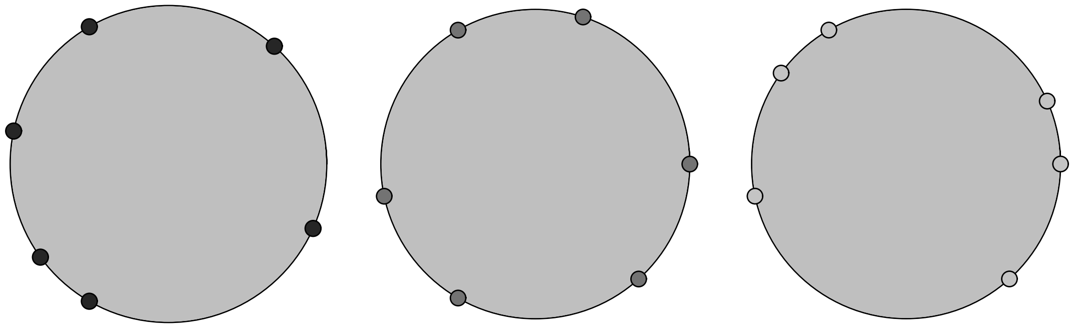

Figure 3. Some sets maximizing the sums of distances for the interval metric – $c = 15, d = 6$

As none of these definitions (or others) appears completely satisfactory in our opinion, we will now venture to propose another one.

### 2.2 *An illuminating remark by Ian Quinn*

Discussing a general typology of chords (or pc-sets), Ian Quinn noticed ( [17], 3.2.1) that what he calls ‘generic prototypes’ are the ME sets, and that they share an extremal property in terms of Fourier ‘weight’[1]. This is what we will now adopt as a definition; Quinn’s impressive survey and classification of the landscape of all chords was not focused exclusively on ME sets, and as his writing voluntarily avoided, to quote him, the ‘stultifying’ quality inherent to dry mathematical generalizations, he left room for a formal proof that this definition is equivalent to the traditional ones (we will prove the following definition is equivalent to the classicical description, up to and including the formula with $J$ functions; see [7] and [10] for equivalence between all other definitions).

Moreover, and this is in itself justification enough for what follows, many properties of ME sets will now appear obvious from this starting point. Finally, the only quantity involved is $|\mathcal{F}_A|$, the ‘chord quality’ or ‘weight’ which is, as we have seen, in many ways the most natural musical invariant for pc-sets.

## 3 A Lewinesque definition of ME sets

### 3.1 *Definition and properties*

Definition 3.1 *The pc-set $A \subset \mathbb{Z}_c$, with cardinality $d$, is a ME set, if the number $|\mathcal{F}_A(d)|$ is maximal among all pc-sets with cardinality $d$:*

$$\forall A' \subset \mathbb{Z}_c, \quad \mathrm{Card}\, A' = d \quad \Rightarrow \quad |\mathcal{F}_A(d)| \geq |\mathcal{F}_{A'}(d)|$$

As the number of pc-sets is finite, a solution must exist. Note that this is closely related to the interval vector of $A$, as we have seen above: $\sqrt{\mathcal{F}(IC_A)(d)} = |\mathcal{F}_A(d)|$. This is a good sign, as it relates the DFT to the mutual intervals in the chord.

[2]But all *strictly* convex distance functions on the unit circle will give maximums on the same pc-sets, which are the ME sets, as shown in [12]. Nonetheless, such a distance (like the chordal distance, length of the line segment between two points of the circle) has little musical meaning.

[1]“ We note that generic prototypicality may be interpreted as maximal imbalance on the associated Fourier balance – at least to the extent that a generic prototype tips its associated Fourier balance more than any other chord of the same cardinality possibly can”.

From the invariance of quantity $|\mathcal{F}_A|$ under musical operations (see theorem 1.4) follows without further ado

THEOREM 3.2 *Transposition, inversion of a ME set still yields a ME set.*

Almost as straightforward, from computation of the DFT of the complement (lemma 1.2) we get

THEOREM 3.3 *The complement of a ME set is a ME set.*

*Proof* For any subset $A$

$$|\mathcal{F}_{\mathbb{Z}_c\setminus A}(c-d)| = |\mathcal{F}_{\mathbb{Z}_c\setminus A}(-d)| = |-\overline{\mathcal{F}_A(d)}| = |\mathcal{F}_A(d)|$$

So the one is maximal whenever the other is, e.g. $A$ is a ME set iff $\mathbb{Z}_c \setminus A$ is. □

Also notice that this definition addresses the *unordered* pc-set.

### 3.2 *The regular polygon case*

It is certainly desirable that in the case when $d \mid c$, the solutions of the above maximization problem be the regular $d-$polygons. Such is the case :

LEMMA 3.4 *For any pc-set $A$ with $d$ elements, any $t \in \mathbb{Z}_c$, $|\mathcal{F}_A(t)| \leq d$.*[1]

Hence any regular $d-$polygon $A = \{0, c/d, 2c/d \dots\}$ or any translate, which verifies

$$|\mathcal{F}_A(d)| = \left|\sum_{k=0}^{d-1} e^{-2i\pi\, d\, kc/d}\right| = |\sum_{k=0}^{d-1} 1| = d$$

is indeed a ME set. When $d \mid c$, the reciprocal is easy:

THEOREM 3.5 *If $|\mathcal{F}_A(d)| = d = \operatorname{Card} A$, then $A$ is a regular polygon.*

This will be indeed a special case of the general computation, but it helps to understand what is going on. This theorem is clearly stated in [17] under a different form.

*Proof* By MINKOWSKI's inequality, the sum $|\sum_{k\in A} e^{-2i\pi k\, d/c}|$ is less than $\sum_{k\in A} |e^{-2i\pi k\, d/c}| = \sum 1 = d$, with equality if and only if all the exponentials share the same direction, i.e. all $-2\pi k\, d/c$ are equal modulo $2\pi$; this occurs only when the $k'd/c - kd/c$ is an integer for all $k, k' \in A$, that is to say the $k' - k$ are multiples of $c/d$, which proves that $A$ is a subset of a regular $d-$gon (mutual angles being multiples of $2\pi(c/d)/c = 2\pi/d$). As $A$ contains $d$ different points, it is the whole $d-$gon. □

All the exponentials in the formula for $\mathcal{F}_A(d) = \sum e^{-2i\pi kd/c}$ are superimposed, pulling in exactly the same direction, like a Tug of War: see figure 4.

This exemplifies that the above definition aims at looking for the best approximation to a regular polygon — obviously it will be only an approximation when $d$ does not divide $c$, for instance there is no regular heptagon inside the 12 notes universe. Indeed the solution (the major scale $A$ =(0 2 4 5 7 9 11) or any translate thereof) achieves $|\mathcal{F}_A(7)| = 2 + \sqrt{3} \approx 3.73$, still far from the unattainable value 7, but still the largest value possible.

### 3.3 *The chromatic clusters and their DFT's*

As a useful stepping stone, we will characterize chromatic clusters, that is to say bunches of consecutive notes.

[1]Furthermore it can be proved that $|\mathcal{F}_A(d)| \leq \inf(d, c-d)$.

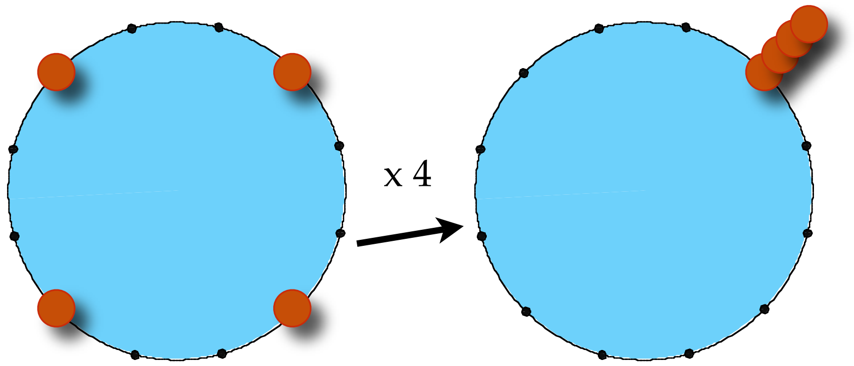


Figure 4. All exponentials superimposed

Theorem 3.6 (Huddling lemma)
*$A$ is a chromatic cluster, i.e. $A \in \mathbb{Z}_c$ is some translate of $\{1, 2 \dots d\}$ if, and only if, $|\mathcal{F}_A(1)|$ has maximal value among all $d-$subsets of $\mathbb{Z}_c$.*

Incidentally there is a formula for this maximal value, namely

$$\mu(c, d) = \frac{\sin((2d+1)\pi/c)}{2\sin(\pi/c)} - \frac{1}{2}.$$

This gets close to $d$ when $c$ is much larger than $d$.

*Proof* Unless it is nil (and hence not a maximum), the sum

$$\mathcal{F}_A(1) = \sum_{k \in A} e^{-2i\pi k/c}$$

of $d$ unit vectors has some direction $\vec{u}$ i.e. $e^{\theta_0}$. Informally, though the $e^{-2i\pi k/c}$ cannot be superimposed as in figure 4, we want them to 'huddle' as near as possible to direction $theta_0$.

Consider for each $k \in A$ the angle $\theta_k = 2\pi k/c - \theta_0$, then the projection of $\mathcal{F}_A(1)$ along direction $\vec{u}$, which is equal to $|\mathcal{F}_A(1)|$ itself, has length $\sum_{k \in A} \cos\theta_k$. As the function cos increases from $-\pi$ to 0 and decreases from 0 to $\pi$, this quantity is maximum when all $\theta_k$'s are as close to 0 as possible. Suppose there is a gap, that is to say some consecutive pair $(j, k)$ of points in $A$ with $\theta_j - \theta_k > 2\pi/c$, meaning $j - k \geq 2$, and no element of $A$ in between. Let us say for instance that $\theta_0 \leq \theta_k < \theta_j$. Now replacing $j$ by $j - 1$ in $A$ replaces $\theta_j$ by $\theta_j - 2\pi/c$ which is closer to 0 and hence has a greater cosine. So $\sum_{k \in A} \cos\theta_k$ was increased, and even though the direction of $\mathcal{F}_A(1)$ will have changed in the process, the new $A$ will have a greater value for $|\mathcal{F}_A(1)|$ (which is longer than its projection in any direction). Hence the original $|\mathcal{F}_A(1)|$ was not maximal because of the presence of gaps. We have thus proved that in order to have a maximal $|\mathcal{F}_A(1)|$, it is necessary to have all elements of $A$ consecutive. Conversely, all such sets are translates of one another and hence give the same value for $|\mathcal{F}_A(1)|$. □

*Remark 1* This means in effect that in order to increase the sum, one moves the points 'inwards' until the set $A$ is 'without holes'. This iterative idea is fairly similar to the proof that the extreme potential configurations in the Ising model are reached for ME sets in [12].

Later on, it will be useful to understand that lemma as compelling the farthest, extreme points of $A$ to move 'inside' as much as possible.

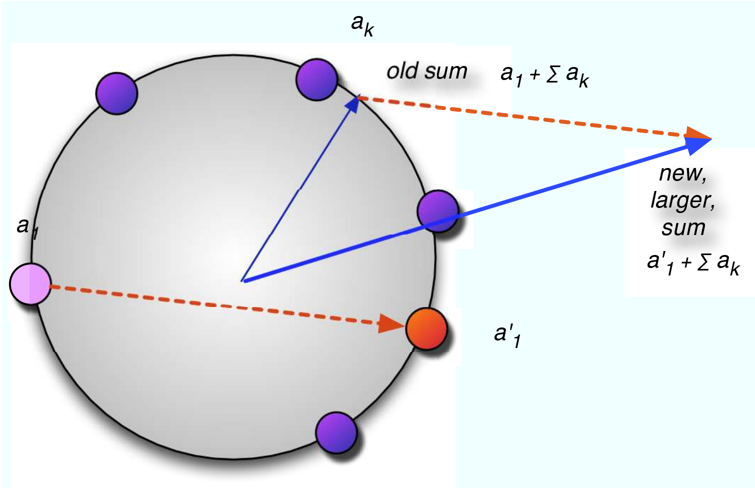


Figure 5. Moving one point 'inside' increases the sum

### 3.4 *These are the usual ME sets*

We now move towards understanding the general case from this definition.

Recall we aim at getting the $e^{-2i\pi\, k\, d/c}, k \in A$, as close as possible, so that their sum gets maximum, as mentioned in the textbook case of a regular polygon.

#### 3.4.1 *The simplest case:* $\gcd(c, d) = 1$.

In this paragraph, dealing with the original ME sets in the restricted sense of [8], we assume $c$ and $d$ coprime ($\gcd(c, d) = 1$): then $x \mapsto d\,x$ is bijective in $\mathbb{Z}_c$, as $d$ has a multiplicative inverse $f = d^{-1} \mod c$.

But $|\mathcal{F}_A(d)| = |\sum_{k\in A} e^{-2i\pi\, k\, d/c}| = |\mathcal{F}_{dA}(1)|$, and hence, $|\mathcal{F}_A(d)|$ is maximum if and only if $|\mathcal{F}_{dA}(1)|$ is; as we have just seen, this means that $dA \in \mathbb{Z}_c$ is a chromatic cluster:

$$\exists a \in \mathbb{Z}_c, dA = a + \{1, 2, \ldots d\}$$

Multiplying by $f = d^{-1}$, we get that $A$ is (up to translation) *generated* by $f$:

Theorem 3.7 *ME sets with $d$ elements, $d$ coprime with $c$, are generated by $f$, the inverse in $\mathbb{Z}_c$ of the cardinality $d$:*

$$A = f\,a + \{f, 2f, 3f \ldots d\,f\} \mod c$$

The typical example is the major scale, generated by a cycle of fifths e.g. seven semi-tones.

*Remark 2* Notice that, as the inversion $-A$ of a ME set $A$ is also a ME set, it is also possible to use generator $f' = -d^{-1}$. For instance, the major scale is also generated by fourths.

*Remark 3* Thm. 5 means that the points in $d\,A$ must be as close as possible, which is also clearly stated in [17] under a dual form[1].

We have covered the original ME sets defined by [8], and the regular polygons. It remains to check the third and last type of ME set, i.e. the case $d \neq \gcd(c, d) > 1$.

[1] " The best the chord can do is to have pcs gathered in adjacent pans, so that the arrows point in approximately the same direction" [17].

**3.4.2 *How it works when $(c, d) > 1$.*** In effect, we have already settled the subcase $1 < d \mid c$: the maximum value of $|\mathcal{F}_A(d)|$ is $d$, and as proved above it is obtained for regular $d-$gons and only for them. So we might now assume that $d$ is not a divisor of $c$. From now on, let $m = \gcd(c, d) > 1, c' = c/m$ and $d' = d/m$ where $m = (c, d)$: so $c'$ and $d'$ are coprime. The map $x \mapsto d\,x \mod c$ is no longer one to one:

LEMMA 3.8 *The image of the group morphism $\varphi_d : x \mapsto d\,x$ from $\mathbb{Z}_c$ into itself, is the subgroup $m\,\mathbb{Z}_c$. This subgroup is isomorphic to $\mathbb{Z}_{c'}$. The kernel of $\varphi_d$ is the subgroup $c'\,\mathbb{Z}_c$, with $m$ elements.*

*Proof* The image of $\varphi_d$ is a cyclic subgroup of $\mathbb{Z}_c$ (generated by $d$... among others). Let $x \in \ker\varphi_d$, i.e. $d\,x = 0$ in $\mathbb{Z}_c$: this means exactly that $c' \mid x$ in $\mathbb{Z}$, as $c \mid d\,x \iff c' \mid d'\,x$, which implies $c' \mid x$ as $(c', d') = 1$.

This characterizes $\varphi_d$: its kernel $c'\,\mathbb{Z}_c$ has $c/c' = m$ elements and hence its image has $c/m = c'$ elements, since $\mathbb{Z}_c/\ker\varphi_d \approx \mathrm{Im}(\varphi_d)$. Or more directly, the kernel being $c'\,\mathbb{Z}_c$, the image is isomorphic to $\mathbb{Z}/c'\mathbb{Z} = \mathbb{Z}_{c'}$. $\square$

So any element of $d\,\mathbb{Z}_c = m\,\mathbb{Z}_c$ has a fiber with $m$ elements, meaning each point in the image $\mathrm{Im}(\varphi_d) = m\,\mathbb{Z}_c$ is obtained exactly $m$ times:

$$dy \equiv dx \mod c \iff y - x \in \ker\varphi_d \iff y = x + kc', k = 0 \dots m-1$$

Here is an example: have $c = 12, d = 8$. Then $m = 4$. The image of $\varphi_d$ is the subgroup $m\,\mathbb{Z}_c = \{0, 4, 8\}$ with $c' = 3 = 12/4$ elements, each of which has $m = 4$ preimages – the preimages of 4, say, are 1, 4, 7, and 11.

Let us now extend the idea of the 'huddling lemma' farther.

**3.4.3 *The multiset $A'$.***

LEMMA 3.9 *For any pc-set in $\mathbb{Z}_c$, $|\mathcal{F}_A(d)| = |\sum_{k\in A} e^{-2i\pi(k\,d)/c}| = |\mathcal{F}_{A'}(1)|$ where $A' = dA$, with the proviso that $A'$ is a* multiset *: each of its elements has a multiplicity (at most $m$, as seen above).*

A multiset in $\mathbb{Z}_c$ can be modeled as a list of multiplicities of elements of $\mathbb{Z}_c$, or equivalently as a map from $\mathbb{Z}_c$ to non negative integers. For a traditional set, the values of the map would be either 0 or 1, for a multiset it can be any integer. The interval function mentioned above can also be seen as a multiset. Here we will denote ${}^m a$ if point $a$ has multiplicity $m$, and list the elements of $A'$ as the ${}^m k$'s with $m = \mathrm{Card}(\varphi_d^{-1}(k) \cap A)$, the number of preimages of $k$ in $A$.

For instance in the extreme case of a regular polygon, $d \mid c$ and $A'$ is a single point $k$ repeated $m = d$ times: $A' = \{{}^d k\}$.

As a more general example, consider the octatonic scale $A = \{0, 1, 3, 4, 6, 7, 9, 10\}$: there $A' = 8A \mod 12 = \{0, 8, 0, 8, 0, 8, 0, 8\} = \{{}^4 0, {}^4 8\}$.

*Until the end of this section we assume that $A$ is a ME set with $d$ elements.*

For clarity, let $A''$ be the ordinary set with the same points as multiset $A'$, i.e. it is $A'$ but with multiplicities are cut down to 1. Now in accordance with our definition, we want $|\mathcal{F}_A(d)| = |\sum_{a\in A''} m(a)e^{-2i\pi a/c}|$ to be as big as possible. Remembering the 'huddling lemma', this means that *the elements $a$ of $A'$ are as close together as possible.* Notice that the ambiant universe is now $d\,\mathbb{Z}_c = m\,zc$, and no longer $\mathbb{Z}_c$. We prove the following generalization of Thm. 5:

PROPOSITION 3.10 *Let $\mathcal{M}$ be the set of multisets in $m\,\mathbb{Z}_c$ with $d$ elements [counting their multiplicities], with the additional constraint that any element of $A' \in \mathcal{M}$ has multiplicity $\leq m$. Then the maximal value of $S = |\sum_{a\in A''} m(a)e^{-2i\pi\,a/c}|$ where $A' \in \mathcal{M}$ and $A''$ is as above, is obtained when $A'$ is some translate of $\{{}^m 1, {}^m 2, \dots {}^m d'\}$.*

*Proof* We assume that $A'$ does not have this form, and prove that $S$ is not maximal with the same arguments as in the proof of the huddling lemma, eg 'filling in the holes' increases the length of the sum.

Let again the direction of the sum $\sum_{a\in A''} m(a)e^{-2i\pi\, a/c}$ be $\theta$, i.e. arg $S=\theta$ (if $S=0$ then it is not maximal). Still

$$|S| = \sum_{a\in A''} m(a)\cos(2i\pi\, a/c+\theta_0) = \sum_{a\in A''} m(a)\cos(-\theta_a) = \sum_{a\in A'} \cos(-\theta_a) \quad \text{(with repetitions)}$$

by projection on its own direction $\theta_0$. The angles $\theta_a = -2i\pi\, a/c - \theta_0$ lie between $-\pi$ and $\pi$. Let $a_{max} \in A'$ with $\theta_{a_{max}}$ closest to $\pi$, i.e. the point $e^{-2i\pi\, a_{max}/c}$ is farthest from the direction $\theta_0$ of the overall sum. Symmetrically, let $a_{min} \in A'$ with $\theta_{a_{min}}$ closest to $-\pi$. For clarity let us assume $-c/2 < a_{min} < 0 < a_{max} < c/2$ (translating $A'$ if necessary). By our assumption that $A'$ is not made of $d'$ consecutive points with maximal multiplicity $m$, $a_{max} - a_{min} + 1 > d'$ and there is some point $a_h$ between them with multiplicity less than maximal:

$$a_{min} < a_h < a_{max} \qquad m(a_h) < m$$

(informally there are 'holes'). At least one of $a_{min}, a_{max}$ is farther from $\theta_0$ than $a_h$. Say it is $a_{min}$ for instance.

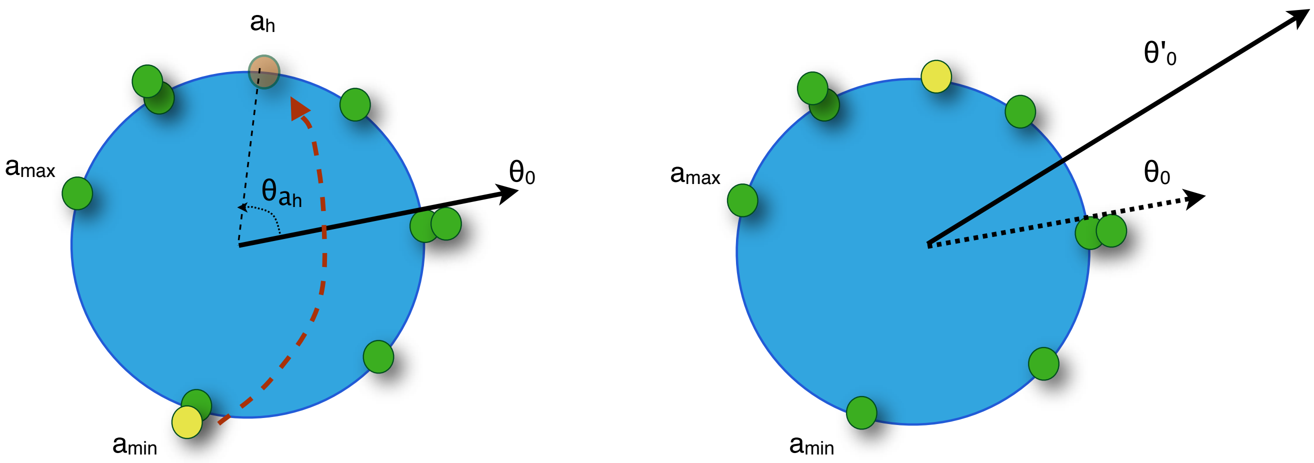


Figure 6. Maximizing the sum on a multiset

Moving one point from position $a_{min}$ to position $a_h$, i.e. decrementing $m(a_{min})$ while incrementing $m(a_h)$, will increase $|S|$ as a $\cos\theta_{a_{min}}$ is replaced by $\cos\theta_{a_h}$ which is greater. Now the direction of the new sum $S'$ will have changed in the process, but its length, being greater that its projection on direction $\theta_0$, must have increased.

We have proved that the sum $S$ is not of maximal length when $A'$ is not a translate of $d'$ consecutive points with multiplicity $m$. Hence this last distribution is maximal. □

As we have several ($d'$) points in multiset $A' = d\,A$, each with multiplicity $m$, it means that set $A$ is periodic (with period $c'$), as the reunion of the sets of all $m$ preimages of elements of $A''$, which are periodic subsets. We have proved

Proposition 3.11 *$A$ is periodic with period $c'$.*

In the example of the octatonic scale, we had $A' = \{^4 0, ^4 8\}$, with preimages 0, 3, 6, 9 for 4 and 1, 4, 7, 11, for 8: $A = \{0,1\} \oplus \{0,3,6,9\} = B \oplus 3\,\mathbb{Z}_{12}$.

Let us introduce for clarity $B$ as the set with elements chosen between 0 and $c'-1$ and $A = B \oplus c'\mathbb{Z}_c$, that is to say we select for each multipoint in $A'$ the one and only preimage in $A$ that is (the residue class of an integer) between 0 and $c'-1$: this is possible as the morphism $\varphi_d$ is bijective when restricted from

$\{0, 1, \ldots c' - 1\}$ to $m\,\mathbb{Z}_c$. Elements of $B$ are actually computed by the canonical morphism from $\mathbb{Z}_c$ to $\mathbb{Z}_{c'}$, i.e. $x \mapsto x \mod c'$.

THEOREM 3.12 *$A$ is a ME set in $\mathbb{Z}_c$ if and only if $B$ is a ME set in $\mathbb{Z}_{c'}$.*

This is the well-known hereditary characterization of type III ME sets in [7], and it rounds up the classification of all ME sets.

*Proof* Actually, computing $d' \times B$ in $\mathbb{Z}_{c'}$ is straightforward and would yield the theorem. But keeping in line with the aim of this paper, we will make use again of the Fourier transform of $B$. From $A = B \oplus c'\mathbb{Z}_c$ and the definition of Fourier transform, we get

$$\mathcal{F}_A(d) = \sum_{k\in A} e^{-2i\pi dk/c} = \sum_{k''\in B} \sum_{\ell=0}^{m-1} e^{-2i\pi d(k''+\ell\, c')/c} = \sum_{k''\in B} e^{-2i\pi d\, k''/c} \sum_{\ell=0}^{m-1} e^{-2i\pi \ell} = m \times \sum_{k''\in B} e^{-2i\pi d'\, k''/c'}$$

so this is (multiplied by the constant $m$) just the DFT of $B \subset \mathbb{Z}_{c'}$, i.e. $\mathcal{F}_B(d')$: this number is of maximal length if and only if $B$ is a ME set in $\mathbb{Z}_{c'}$ (we have already proved this in paragraph 3.7, as $c'$ and $d'$ are coprime). $\square$

*Remark 4* The computation obfuscates the essential obviousness of this result: we have seen that $A$ must be $c'-$periodic, so its DFT in essence has only $c'$ meaningful values, the others are nil. It follows from Prop. 1.5 that $\mathcal{F}_A(k) = 0$ unless $c' \mid k$. So checking out the useless values of $k$, one remains with the DFT of a subset of $\mathbb{Z}_{c'}$, none other than $B$. This is clearly visible on figure 7 with Fourier transforms of the ME set (0 2 4) in $\mathbb{Z}_7$ and its counterpart (0 2 4)$\oplus$ (0 7 14 21) in $\mathbb{Z}_{28}$. This argument seems to us more illuminating than purely algebraic computations, as it enhances the fact that the "characteristic domain" $B$ concentrates its energy in the sense of the huddling lemma, in order for $A$ to do the same.

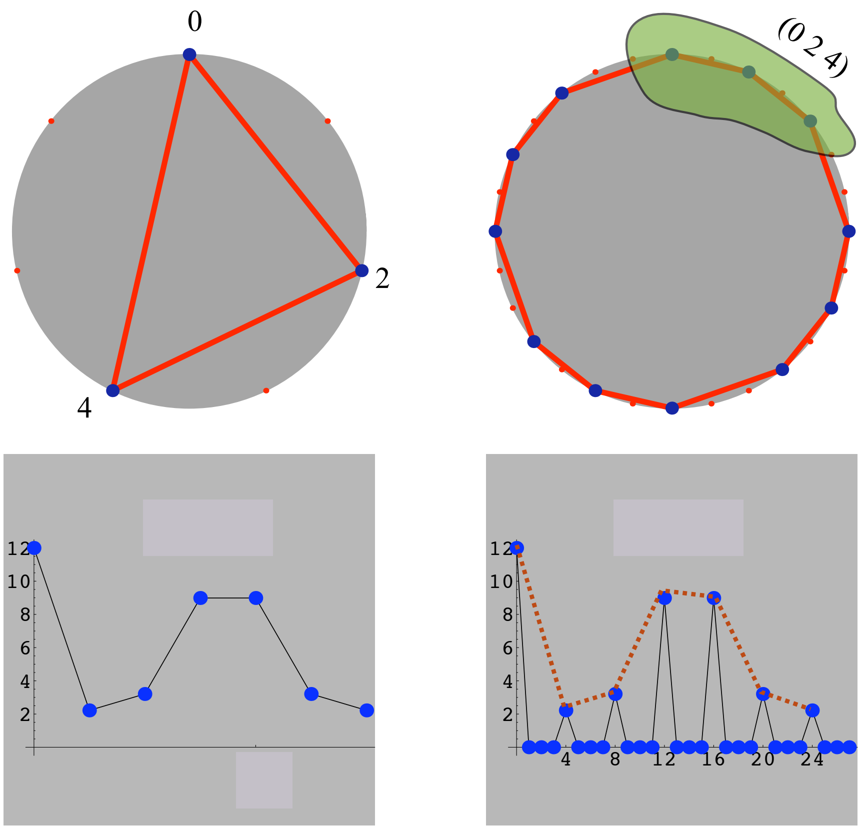


Figure 7. Maximizing for $B$ is maximizing for $A$

We get as a corollary what we might have got by direct computation:

THEOREM 3.13 *A is a ME set of cardinality d in* $\mathbb{Z}_c$ *if and only if the set* $d' \times B = (d' \times A \mod c') \in \mathbb{Z}_{c'}$ *is some translate of* $\{1, 2, \dots d'\} \subset \mathbb{Z}_{c'}$.

We immediately get from there the complete enumeration of ME sets, which stands in all three cases (it is clearly true already in the first two, which may also be regarded as special cases with respectively $m = 1$ and $m = d$).

COROLLARY 3.14 *The number of different ME sets of cardinality d in* $\mathbb{Z}_c$ *is* $c' = c/\gcd(c, d)$ *(the number of different possible B's).*

*All are translates of one another (the group of translations acts transitively on ME sets)*[1].

For each couple $(c, d)$ there is but one translation class of ME sets with $d$ points in $\mathbb{Z}_c$. Henceforth we will denote such a ME set *class* as $\langle c, d\rangle$. An actual ME set will be 'a $\langle c, d\rangle$ ME set'. For example there are exactly three different $\langle 12, 8\rangle$ ME sets, i.e. the octatonic scales.

*Remark 5* Each individual $< c, d >$ ME set is invariant under the $m$ translations of step $c'$ and multiples.

We have seen (1.4) that the inversion operation preserves the class of ME-sets: this means that the inverse of a ME set is one of its translates. Indeed a ME set is its own image under exactly[2] $2 \times m$ operations, $m$ translations and $m$ inversions in the dihedral group T/I of transformations of type $x \mapsto x + \tau$ and $x \mapsto \ell - x$ in $\mathbb{Z}_c$. For instance, inversions $x \mapsto -x, 3 - x, 6 - x, 9 - x$ preserve the above octatonic.

### 3.4.4 *Commutative diagramms.*

*Remark 6*

[to reviewers] I am in doubt about this paragraph: it is not necessary in the flow of the actual paper, but it might be useful for further research. So shall I keep it or not ?

This paragraph is ambling away from the general scope of the paper, since instead of making, hopefully, the notion of ME sets more intuitive, geometric and musical, it introduces rather abstract algebra. We include it nonetheless for at least two reasons:

- Nowadays most musicologists are getting familiar with digraphs, or will, and
- abstraction comes at a cost, but pays off when further developments and/or connections are needed. Though we do not see at present, for instance, how Pairwise Well-Formed Scales could benefit from this Fourier analysis of subsets, a strong formalized approach of the present work could help develop further research on this subject.

As we have established above, a ME set $A$ with $d$ elements in $\mathbb{Z}_c$ is $c'$- periodic. It is built up from its projection : $A \mod c' = B \subset \mathbb{Z}_{c'}$, hence $A = B + c'\,\mathbb{Z}_c$ (this makes sense as the set of preimages of $B$ by the the canonical projection from $\mathbb{Z}_c$ onto $\mathbb{Z}_{c'}$) and as $B \subset \mathbb{Z}_{c'}$ is a ME set with $d'$ elements, $d'$ being coprime with $c'$, $B$ is (up to translation) an arithmetic sequence $B = (b+)f' \times \{1, 2, \dots d'\} \pmod{c'}$. So we can express this compound of multiplications, reductions with different moduli, and isomorphisms ($\approx$) by a couple of commutative diagrams, the second being included in the first:

$$\begin{array}{ccc} \mathbb{Z}_c & \xrightarrow{\varphi_d} & m\,\mathbb{Z}_c \\ {\scriptstyle \text{mod } c'}\big\downarrow & & \approx\big\downarrow{\scriptstyle t\mapsto t/m} \\ \mathbb{Z}_{c'} & \underset{\approx}{\xrightarrow{\varphi_{d'}}} & \mathbb{Z}_{c'} \end{array} \qquad \begin{array}{ccc} A & \underset{\varphi_d}{\longrightarrow} & A' = d\,A \\ {\scriptstyle \text{mod } c'}\big\downarrow & & \big\downarrow{\scriptstyle t\mapsto t/m} \\ B & \underset{\approx}{\xrightarrow{\varphi_{d'}}} & b + \{1, 2, \dots d'\} \end{array}$$

[1]Only when $m = 1$ do we have simple transitivity, i.e. an interval group in the sense of [15].

[2]The stabilizer of any pc-set in T/I, isomorphic to dihedral group $D_c$, is either a cyclic or a dihedral group. For $\langle c, d\rangle$, it is always a $D_m$.

Hence in practice one generates $B$ with a 'cycle of fifths' of $d'$ elements of $\mathbb{Z}_{c'}$, where the generator $f'$ is the inverse (or its opposite, see remark *2*) of $d' \mod c'$: $B = \{f', 2f', \dots d'f'\} \mod c'$; and $A$ is retrieved by adding multiples of $c'$: $A = \{f', 2f', \dots d'f'\} + c'\,\mathbb{Z}_c$. This description of ME sets in the most complicated case finally matches the historical one.

### 3.5 *Expression by way of J functions*

For the sake of completeness we add this technical but quick derivation of all ME sets[1] as values of a $J$ function:

THEOREM 3.15 *Any ME set $A$ may be obtained by way of a $J$ function, i.e.*

$$\exists \alpha \in \mathbb{Z}, \quad A = \{\lfloor \frac{kc+\alpha}{d} \rfloor \mod c, k = 0 \dots d-1\}$$

This vindicates finally our claim that the Fourier definition of ME sets allows to retrieve all the known theory.

*Proof* We compute first in $\mathbb{Z}$, to get the actual ME set it will only remain to reduce modulo $c$. We take $\alpha$ equal to 0, other choices lead to translates of $A$ (and may change the order of elements, see [7]).

We will define $A$ as the sequence $\lfloor \frac{kc}{d} \rfloor$ for $k = 0 \dots d-1$ and we will prove that $A \mod c$ is a $\langle c, d \rangle$ ME set.

We notice first that this is a sequence of identical subsequences, up to translation:

$$\lfloor \frac{(k+d')c}{d} \rfloor = \lfloor \frac{kc}{d} + \frac{d'c}{d} \rfloor = \lfloor \frac{kc}{d} \rfloor + c' \quad \text{where we put again } m = \gcd(c,d), c = c'\,m, d = d',m$$

Hence it is sufficient to study the subsequence of the first $c'$ terms, i.e.

$$B = \{\lfloor \frac{kc}{d} \rfloor,\ k = 0 \dots d'-1\}$$

We claim that the fractional parts of the numbers $\frac{kc}{d} = \frac{kc'}{d'}$ take different values when $k$ runs from 0 to $d'-1$. This is true because $c'$ and $d'$ are coprime:

$$\frac{k'c'}{d'} - \frac{kc'}{d'} = n \in \mathbb{Z} \Rightarrow (k'-k)c' = d'n \Rightarrow d' \mid (k'-k) \Rightarrow k' = k \text{ as } |k'-k| < d'$$

by GAUSS'S theorem. Next notice that

$$0 \leq \frac{kc'}{d'} - \lfloor \frac{kc'}{d'} \rfloor < 1 \Rightarrow 0 \leq kc' - d'\lfloor \frac{kc'}{d'} \rfloor \leq d'-1$$

and integers $kc' - d'\lfloor \frac{kc'}{d'} \rfloor$ will lie between 0 and $d'-1$ when $k = 0 \dots d'-1$.

But all these integers are distinct, as they are multiples of the fractional parts of the $\frac{kc'}{d'}$, which are distinct as established previously.

So $-d'B = \{-d'\lfloor \frac{kc'}{d'} \rfloor,\ k = 0 \dots d'-1\}$, when reduced modulo $c'$, yields **all** different integers from 0 to $d'-1$. This means that $B \mod c'$ is a $\langle c', d' \rangle$ ME set, from Thm. 3.7.

[1]This is the algorithm given in [8] and extended to all cases in [7], but here we get the formula *after* the classification of all ME sets.

As $A$ is made of copies of $B$, namely $A = B \oplus c'\mathbb{Z}_c$, this proves from Thm. 3.13 and the construction discussed above, that $A = \{\lfloor \frac{kc}{d} \rfloor,\ k = 0 \dots d-1\}$ is a $\langle c, d\rangle$ ME set. □

## 4 Generated scales

A nice added feature of this DFT characterization of ME sets is its extension – *mutatis mutandis* – to the general generated scales, of which we have already seen the chromatic clusters.

Definition 4.1 *$A \in \mathbb{Z}_c$ is generated by $f$ if (up to translation as usual) $A = \{f, 2f, 3f \dots d\,f\}$.*

We get chromatic clusters, ME sets with $f = d^{-1}$, or regular polygons with $f = c/d$, in the cases respectively of $f = 1$, $\gcd(c, d) = 1$ or $c = f\,d$. The general idea is that ones switches between generated scales by way of affine maps; and this does not change the Fourier coefficients, but just permutes them. We have used this when writing $\mathcal{F}_A(d) = \mathcal{F}_{d\,A}(1)$. Unfortunately, owing to the existence of divisors of zero in $\mathbb{Z}_c$, we cannot then and there transpose all the results on ME sets to all generated scales, but some interesting results emerge nonetheless:

Definition 4.2 *Let $\|\mathcal{F}_A\|$ be the maximal value of **all** Fourier coefficients:*

$$\|\mathcal{F}_A\| = \max_{t \in \mathbb{Z}_c, t \neq 0} |\mathcal{F}_A(t)|$$

*and let $\|\mathcal{F}_A\|^*$ be the maximal value of Fourier coefficients with **invertible**[1] indexes:*

$$\|\mathcal{F}_A\|^* = \max_{t \in \mathbb{Z}_c^*} |\mathcal{F}_A(t)|$$

We consider two different cases:

Theorem 4.3
*Let $\mu(c, d)$ be the value of $|\mathcal{F}_B(d)|$ for any $\langle c, d\rangle$ ME set $B$, that is to say $\mu(c, d) = |\mathcal{F}_{1,2\dots d}(1)|$.*
*A pc-set $A$ with $d$ elements is generated by some $f$* coprime *with $c$, if and only if $\|\mathcal{F}_A\|^* = \mu(c, d)$.*

*Proof* Direct sense: $f^{-1}A$ is by assumption a chromatic cluster, hence $|\mathcal{F}_A|$ reaches $\mu(c, d)$ as

$$|\mathcal{F}_A(f^{-1})| = |\mathcal{F}_{f^{-1}A}(1)| = \mu(c, d).$$

This is, as we have seen, the value of $|\mathcal{F}_A(B)(d)|$ when $B$ is any $\langle c, d\rangle$ ME set.

Conversely, let $\|\mathcal{F}_A\|^* = |\mathcal{F}_A(t_0)| = |\mathcal{F}_{t_0\,A}(1)|$ and assume this is the maximal possible value $\mu(c, d) = |\mathcal{F}_{\{1,2\dots d\}}(1)|$. Hence $|\mathcal{F}_{t_0\,A}(1)|$ is maximal, and $t_0\,A$ is a cluster (with $d$ consecutive elements). As $t_0$ is coprime with $c$, hence invertible mod $c$, we get eventually

$$A = t_0^{-1}\{1, 2 \dots d\} \qquad \text{up to translation, i.e. } A \text{ is generated.}$$

□

Please note that $\|\mathcal{F}_A\|^*$ can never exceed $\mu(c, d)$. In other words,

Corollary 4.4 *Maximal values of $\|\mathcal{F}_A\|^*$ pertain to generated scales.*

For instance, scale (0 2 4)= $A \subset \mathbb{Z}_{11}$ is certainly generated; this appears when considering the value $|\mathcal{F}_A(6)| = 2.7287 = \mu(11, 3)$, maximal.

[1]The group $\mathbb{Z}_c^*$ is made up of the classes of integers coprime with $c$.

Unfortunately but interestingly, some generated scales (with generators non coprime with $c$) cannot be characterized in that way, e.g. $A = \{0, 2, 4, 6, 8\} \subset \mathbb{Z}_{12}$: then $\|\mathcal{F}_A\|^* = 1 < 2.73 = \mu(12, 5)$. We can partially reach these scales with the following result, looking for the absolutely maximal value possible:

THEOREM 4.5 *If a scale (or pc-set) $A$ with $d$ elements checks the condition $\|\mathcal{F}_A\| = d$, then $A$ is part of a regular polygon with $c' = c/\gcd(c, d)$ elements.*

*Proof* We have by assumption $|\mathcal{F}_A(t)| = d$ for some $t \neq 0$, but as seen in theorem 3.5, this means that multiset $d\,A$ is a single point with multiplicity $d$. So $\varphi_d : x \mapsto d\,x$ is not one to one, meaning $m = \gcd(c, d) > 1$, and that $A$ is a subset of the preimages of a single element of $m\,\mathbb{Z}_c$. As seen when studying $\varphi_d$, these preimages form a $c'$−polygon. □

For instance, the chunck of whole tone scale $A = \{0, 2, 4\} \subset \mathbb{Z}_{12}$ is certainly not Maximally Even, but it is generated, and this appears when computing $\mathcal{F}_A(6) = 3$, clearly an unbeatable value (notice $|\mathcal{F}_A(3)| = 1 < 3$).

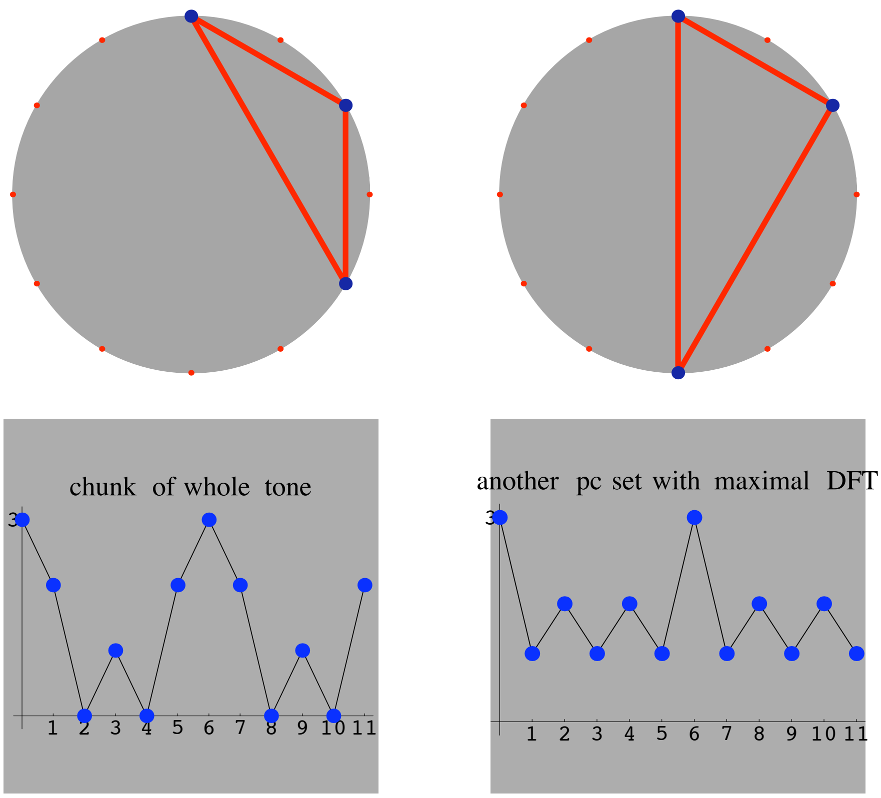


Figure 8. DFT of (0 2 4) and (0 2 6) modulo 12 share maximal value in 6

Notice that, although this includes generated scales, other cases are possible: $C = \{0, 2, 6\}$ also checks $\mathcal{F}_C(6) = 3$. This includes the 'secondary' and many ternary 'prototypes'[1] in [17], 2.4, as it seems thazt Quinn had noticed. This class of maximal pc-sets includes generated scales, but is somewhat wider.

## 5 Chopin's theorem

[1](0 2 6) mod 10 falls under the last theorem, but not (0 1 2 3 4 5 7 8 9).

This section is just an extension of remark *2* above, that either $f' = d'^{-1}$ or its opposite will generate a $\langle c', d' \rangle$ ME set. This has a consequence on complementary ME sets classes: as $\gcd(c, c-d) = \gcd(c, d) = m$, when one replaces $d$ by $c-d$, one gets the same $c'$, and replaces $d'$ by $\dfrac{c-d}{m} = c' - d' \equiv -d' \mod c'$; hence

Lemma 5.1 *A same generator $f'$ can be used for the construction of both $\langle c, d \rangle$ and $\langle c, c-d \rangle$ ME sets.*

The interesting[1] question of the set of **all** generators of a scale (not only for ME sets) is to be elucidated in [2].

For instance, the fifth $f' = f = 7$ generates both the pentatonic and the major scales, when $c = 12$. For, say, $c = 20$ and $d = 8$, one gets $m = 4, d' = 2, c' = 5, f' = 3$ and the generated ME sets with 8 and 12 elements are $\{0, 3\} \oplus \{0, 5, 10, 15\}$ and $\{0, 3, 6, 9\} \oplus \{0, 5, 10, 15\} = \{0, 3, 1, 4\} \oplus \{0, 5, 10, 15\}$. More generally,

Theorem 5.2 *Let $1 < d \leq c/2$; then any given $\langle c, c-d \rangle$ ME set contains several (exactly $c' - 2d' + 1$) $\langle c, d \rangle$ ME sets. In short, the complement of a ME set* contains it *(or the reverse) – up to transposition of course.*

*Proof* A $\langle c, d \rangle$ ME set is constructed by truncating to just $d'$ consecutive values the sequence $\{f', 2f', \ldots (c' - d')f'\} \mod c'$, which generates (adding up $c' \mathbb{Z}_c$) the given $\langle c, c-d \rangle$ ME set $A$. This can be done in precisely $c' - 2d' + 1$ ways.

From there, as seen above, it suffices to add $c' \mathbb{Z}_c$ to get *both* whole ME sets, since $c'$ is the same for $d$ and $c - d$, preserving the inclusion relation all the time. □

We would like to baptize this result Chopin's theorem in reference to the Etude op 10 N°5 (9) where the right hand plays the pentatonic (black keys only) while the left hand wanders through several keys, G flat and D flat major for instance. This result has been observed (especially in this pentatonic $\subset$ major scale case) and commented[2] although perhaps it has not been stated and proved as a quality of all ME sets (or, more generally, generated scales).

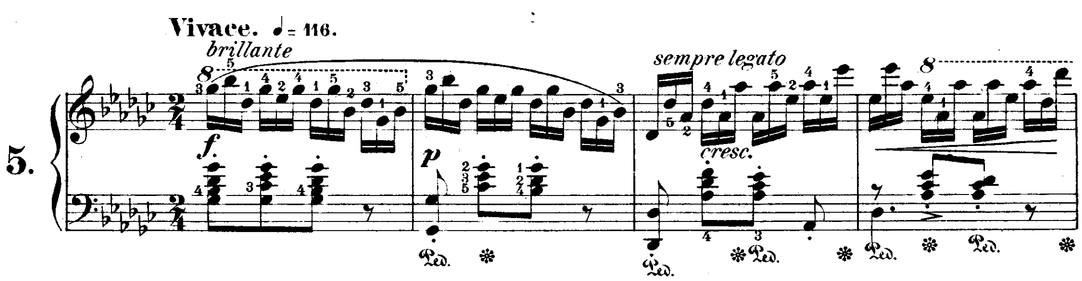


Figure 9. Etude N°5 opus 10, Frédéric Chopin

So David Lewin, who almost invented ME sets as we have seen, might also have originated $Kh-$theory too in one fell swoop.

## 6 Coda

We have examined the definition of the DFT of a pc-set, according to David Lewin. Several interesting features of the pc-set are encapsulated in the absolute value of this function.

Following then Ian Quinn, we were led to advance an original definition of Maximally Even sets, which appears to be geometrical, concise, elegant, and illuminating[3]. We hope that this definition will become a productive one.

[1]In some special cases, there might be more than two generators – or less, eg $d = c/2$.

[2]For instance in [17], 2.3: " all secondary prototypes are Kh-related to one another", which seems to be an equivalent statement to the theorem above.

[3]Though less general than [10] which allows *all* possible strictly convex measures on the unit circle to be chosen indifferently.

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First of all to Ian Quinn who not only spelled out the property which makes the gist of this paper, but also drew our attention, through his comprehensive study of chords landscape, to the impressive advantages of the DFT of chords, and not only ME sets and other 'prototypes'. David Clampitt kindly explained the subtleties of WF scales vs ME sets and most of the history of these fascinating notions. Equally important to the field of 'mathemusical' knowledge is the continued contribution of Jack Douthett (with the late John Clough and other partners). He is still a beacon in the field of ME sets.

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